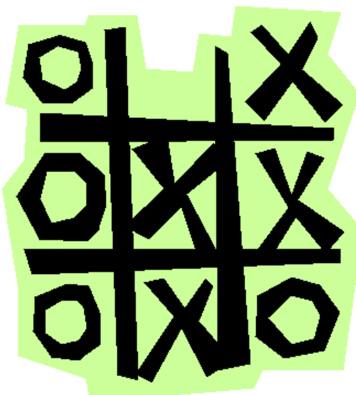
Sequential games

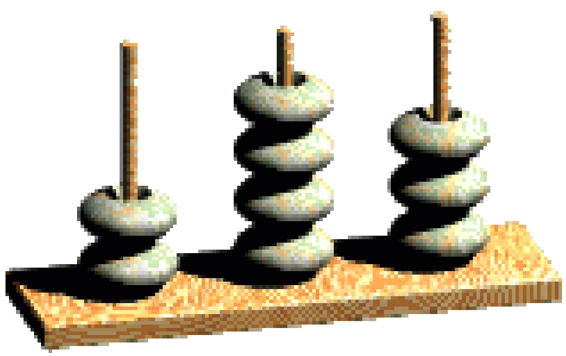
Sequential games

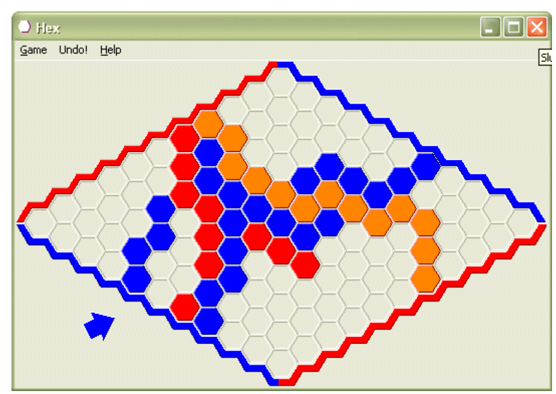
A sequential game is a game where one player chooses his action before the others choose their.

We say that a game has perfect information if all players know all moves that have taken place.

Sequential games





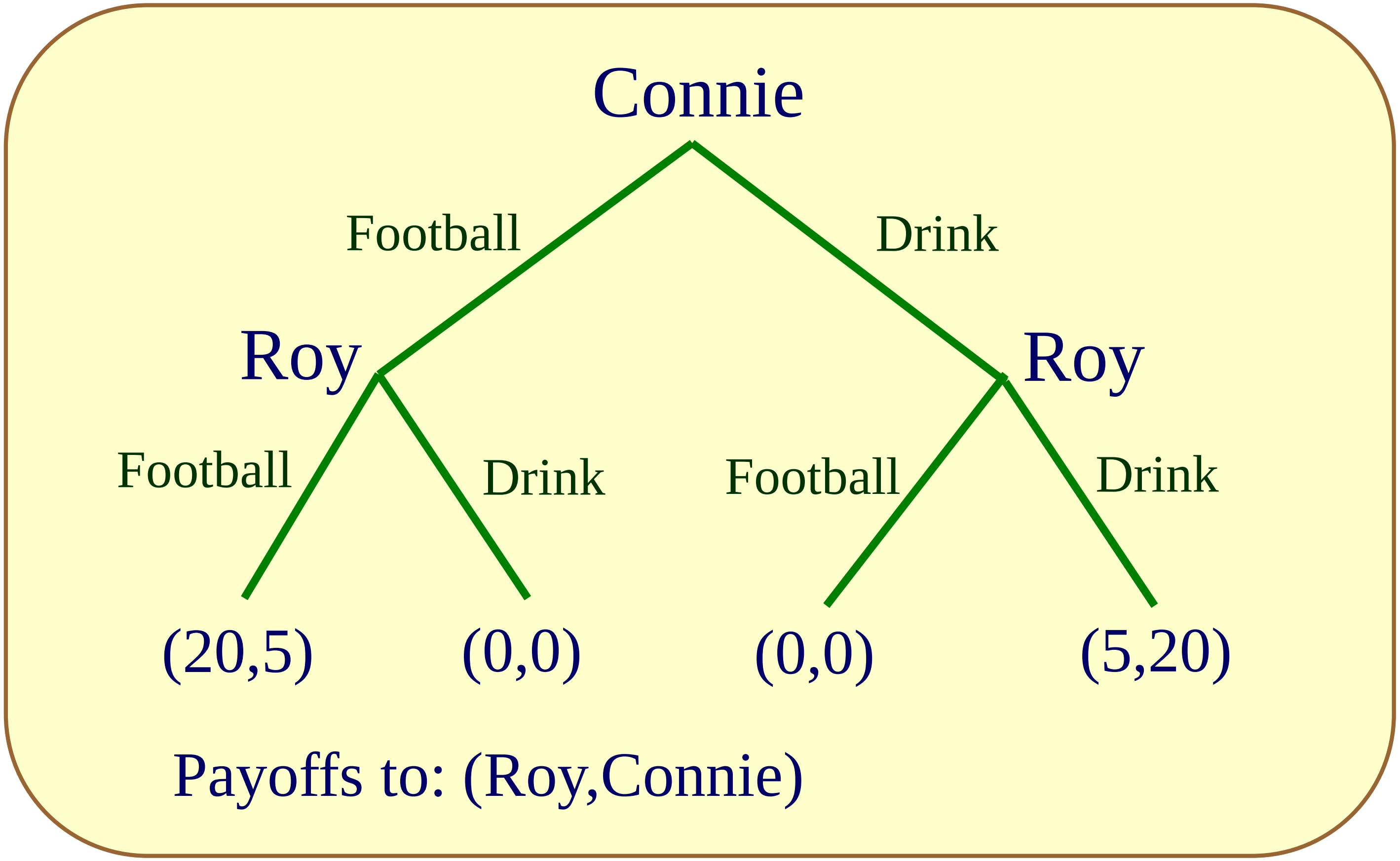


Sequential games

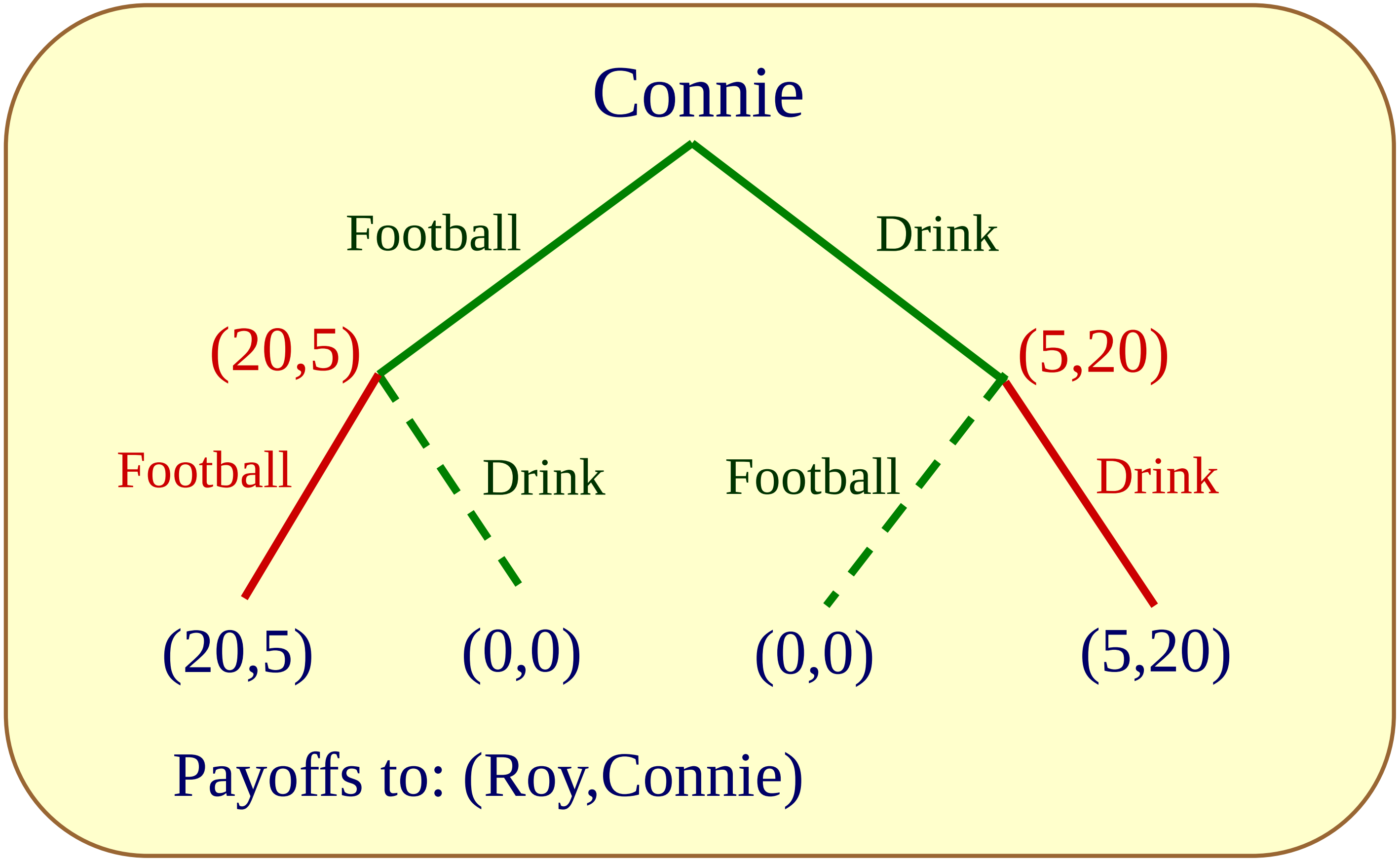
We may play the dating game as a sequential game. In this case, one player, say Connie, makes a choice before the other.

		Connie	
		Football	**Drink**
Roy	**Football**	**(20,5)**	**(0,0)**
	Drink	**(0,0)**	**(5,20)**

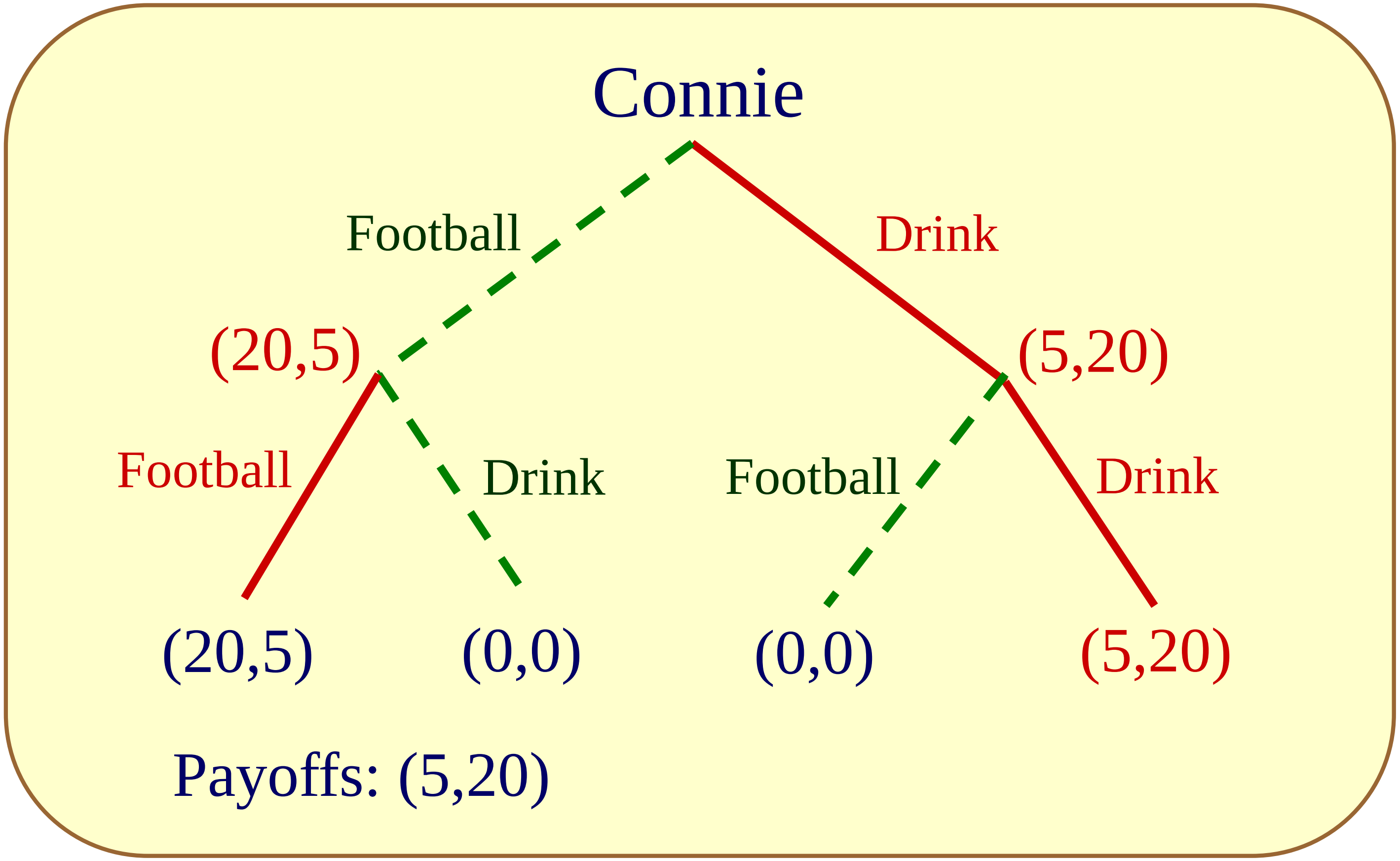
Game tree


Connie
Football
Drink
Roy
Roy
Football
Drink
Football
Drink
(20,5)
(0,0)
(0,0)
(5,20)
Payoffs to: (Roy,Connie)

Backward induction

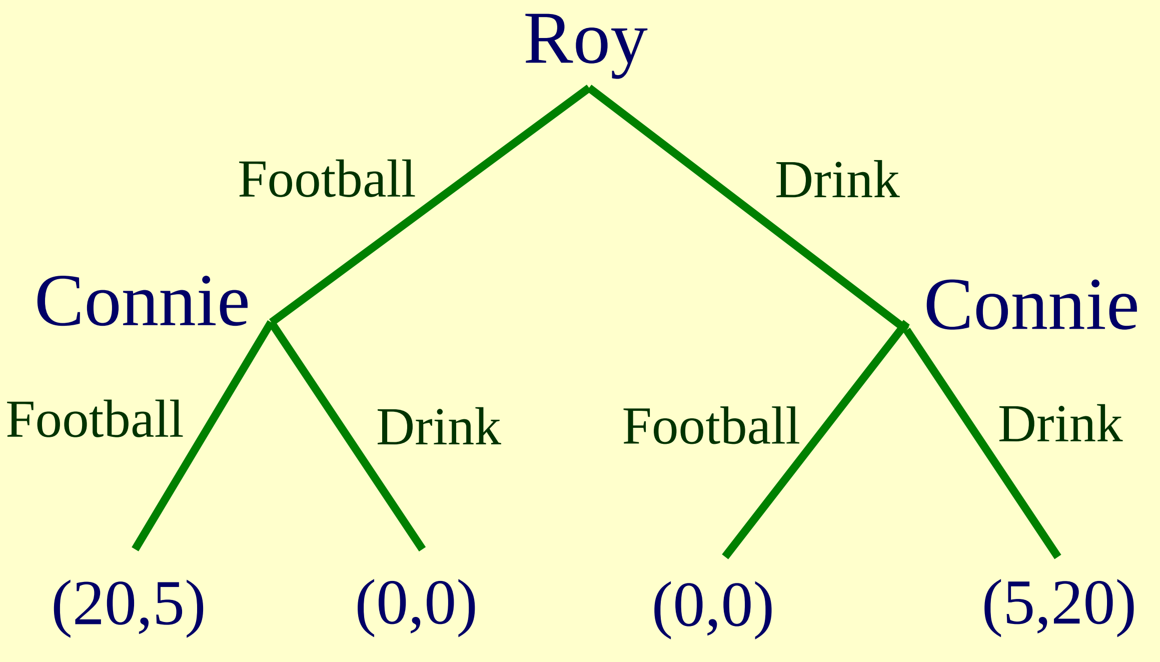


Backward induction



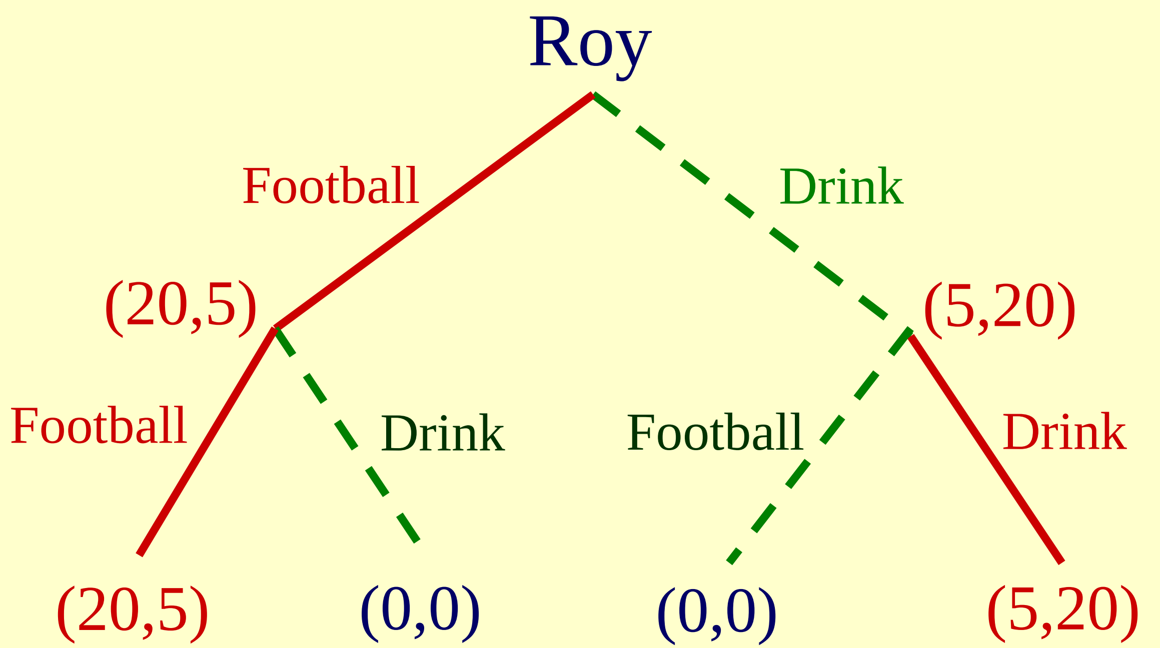
Game tree

Suppose Roy chooses first.



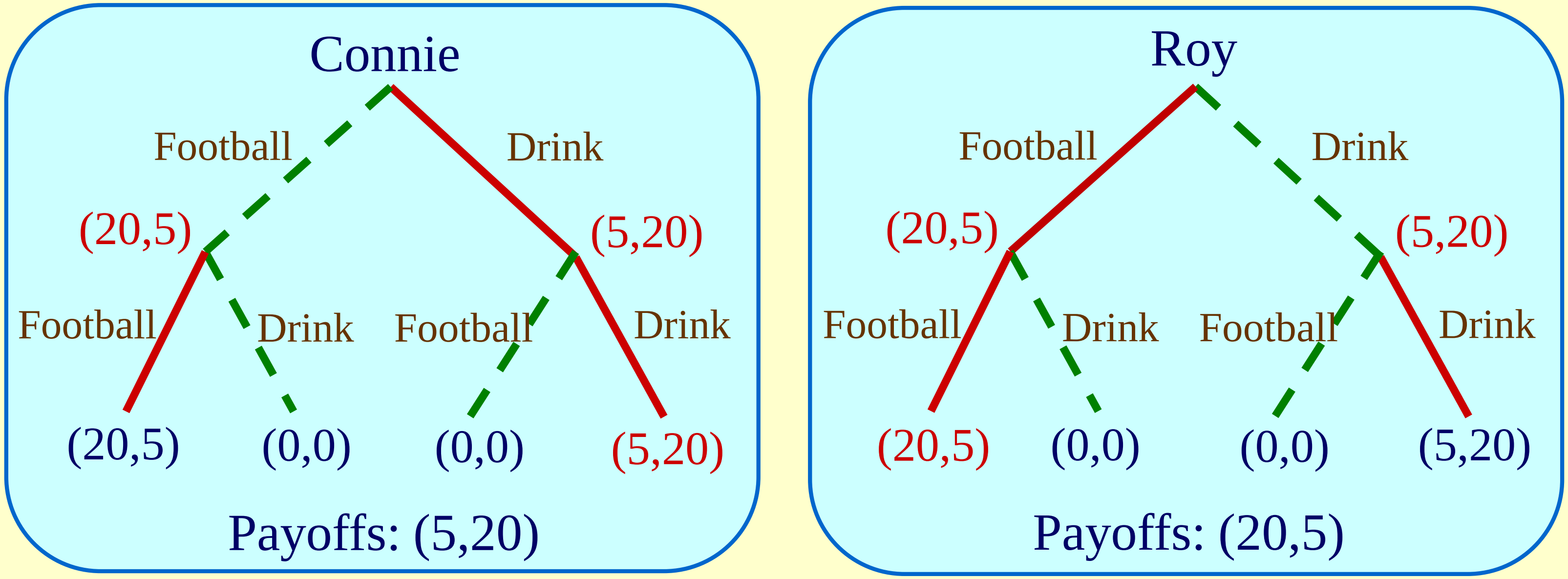
Payoffs to: (Roy,Connie)

Game tree



Payoffs: (20,5)

Game tree



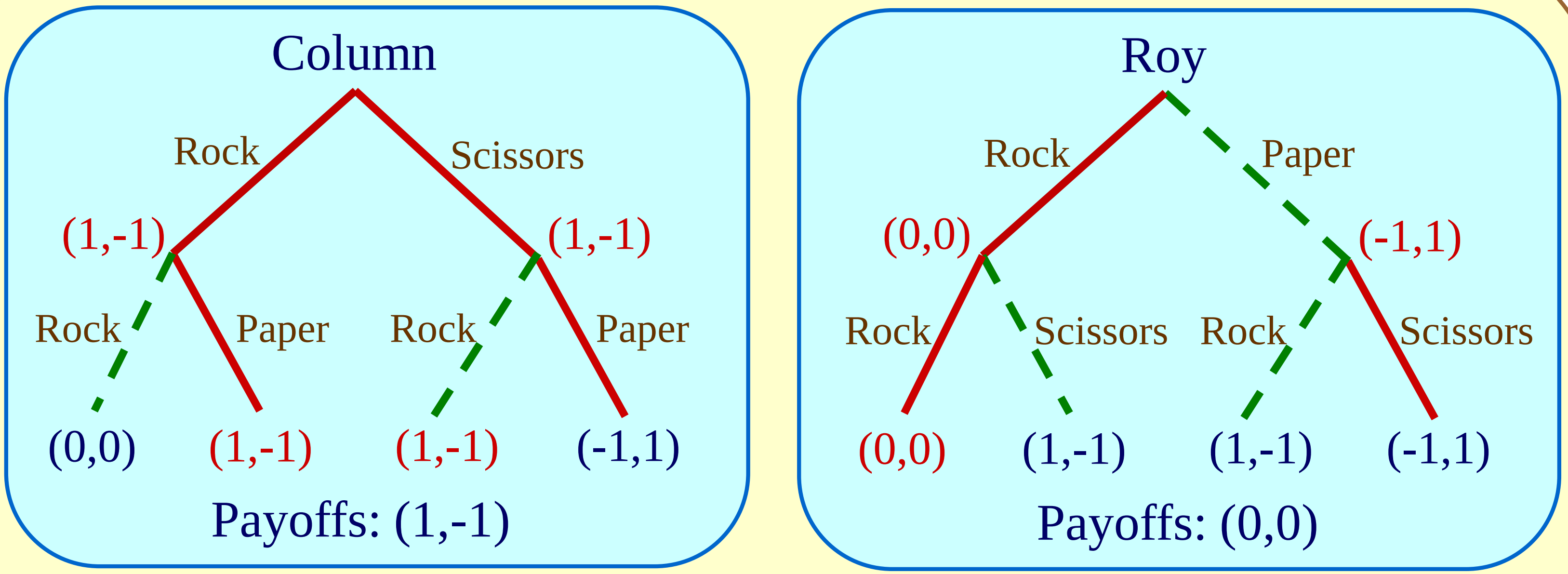
In dating game, the first player to choose has an advantage.

Game tree

Modified rock-paper-scissors

		Column player	
		Rock	Scissors
Row player	Rock	(0,0)	(1,-1)
	Paper	(1,-1)	(-1,1)

Game tree



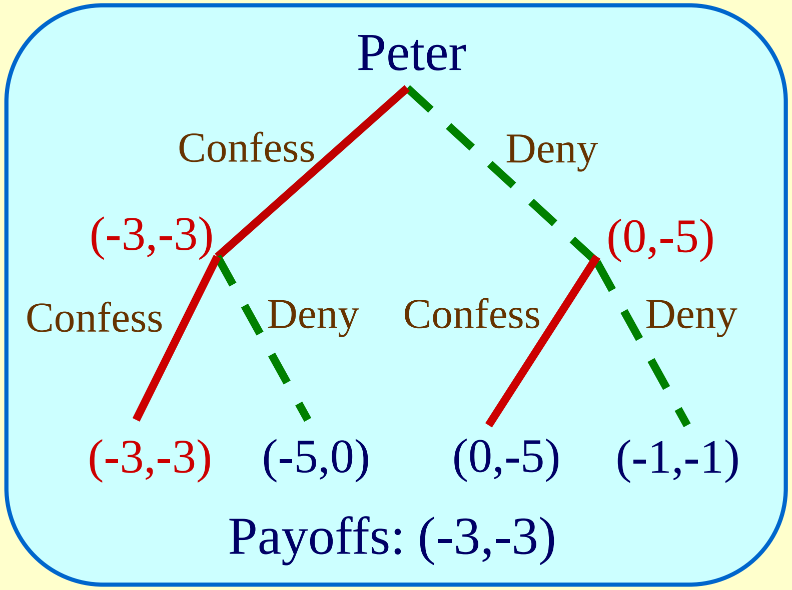
In modified rock-paper-scissors, the second player to choose has an advantage.

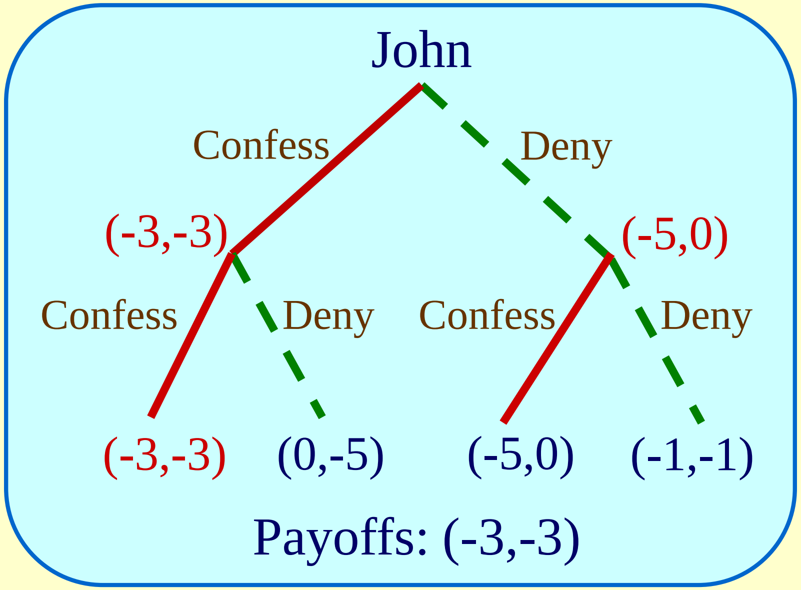
Game tree

Prisoner's dilemma

		Peter	
		Confess	Deny
John	Confess	(-3,-3)	(0,-5)
	Deny	(-5,0)	(-1,-1)

Game tree





In prisoner's dilemma, it doesn't matter which player to choose first.

Combinatorial games

- Two-person sequential game
- Perfect information
- The outcome is either of the players wins
- The game ends in a finite number of moves

Combinatorial games

Terminal position: A position from which no moves is possible

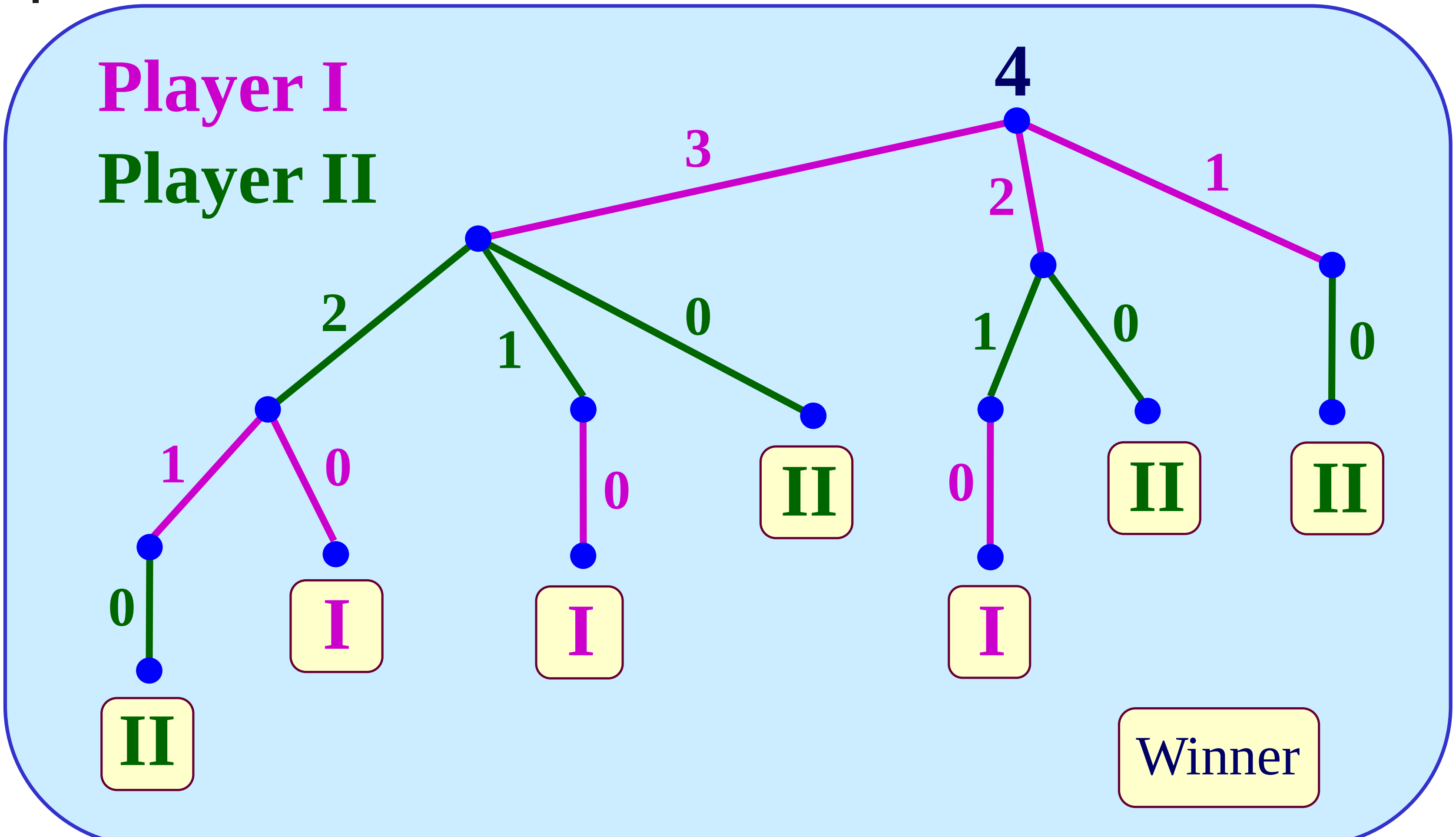
Impartial game: The set of moves at all positions are the same for both players

Normal play rule: The last player to move wins

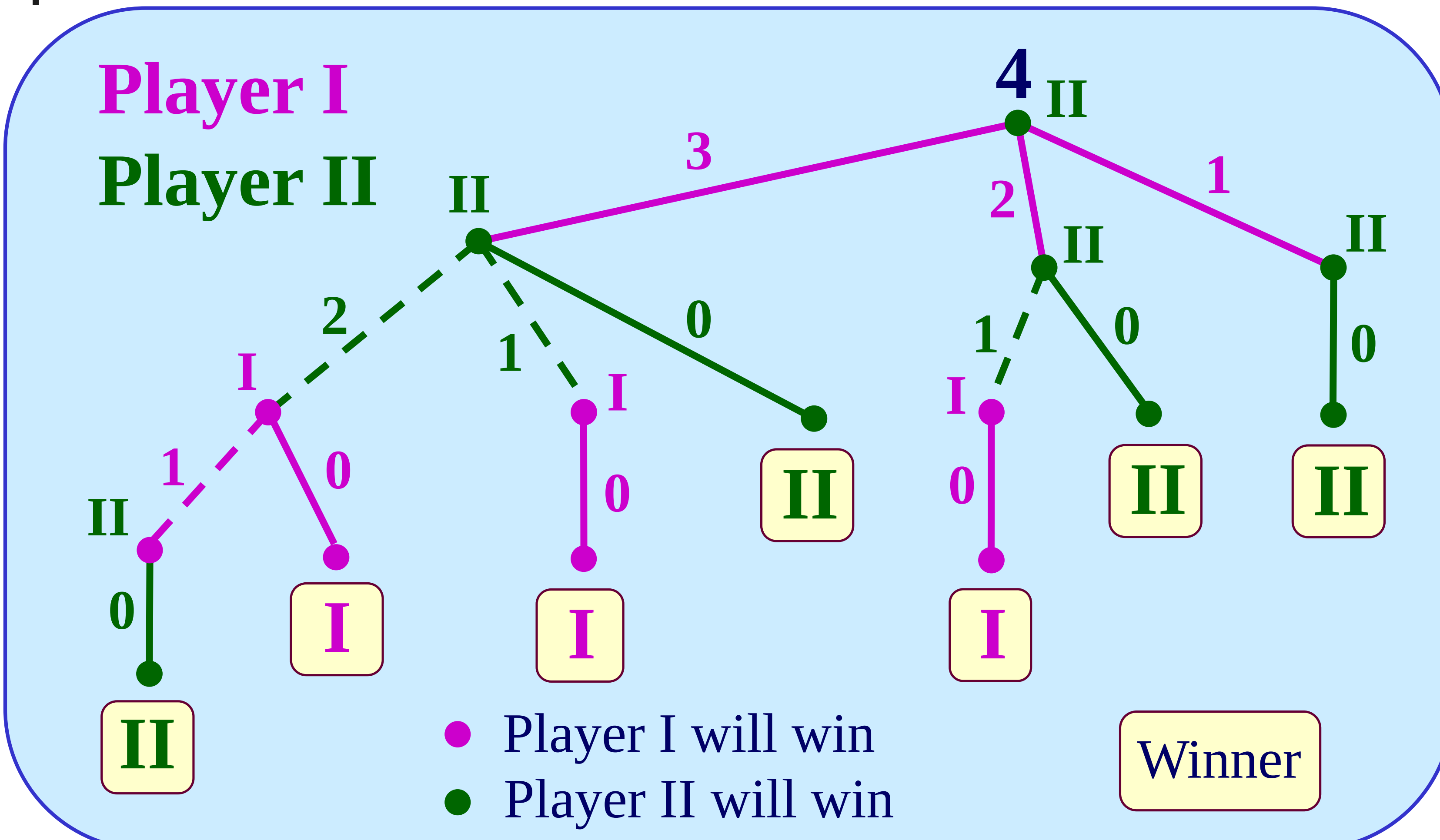
Take-away game

- There is a pile of *n* chips on the table.
- Two players take turns removing 1, 2, or 3 chips from the pile.
- The player removes the last chip wins.

Game tree



Game tree



Player I
Player II
4
II
3
2
1
II
II
II
2
1
0
1
0
0
I
I
I
1
0
0
0
II
II
II
II
0
II
I
I
I
Player I will win
Player II will win
Winner

Take-away game

- When $n = 4$, Player II has a winning strategy.
- More generally when n is a multiple of 4, Player II has a winning strategy.
- When n is not a multiple of 4, Player I has a winning strategy.
- The game tree is too complicate to be analyzed for most games.

Zermelo's theorem

In any finite sequential game with perfect information, at least one of the players has a drawing strategy. In particular if the game cannot end with a draw, then exactly one of the players has a winning strategy.

de Morgan's law

de Morgan's law

$$(A \cap B)^c = A^c \cup B^c$$

$$(A \cup B)^c = A^c \cap B^c$$

de Morgan's law

For logical statements

$$\neg \forall x P(x) \Leftrightarrow \exists x \neg P(x)$$

$$\neg \exists x P(x) \Leftrightarrow \forall x \neg P(x)$$

de Morgan’s law

Example

The negation of

“All apples are red.”

is

“There exists an apple which is not red.”

de Morgan’s law

Example

The negation of

“There exists a lemon which is green.”

is

“All lemons are not green.”

de Morgan's law

More generally

$$\neg \forall x_1 \exists y_1 \cdots \forall x_k \exists y_k P(x_1, y_1, \cdots, x_k, y_k)$$
$$\Leftrightarrow \exists x_1 \forall y_1 \cdots \exists x_k \forall y_k \neg P(x_1, y_1, \cdots, x_k, y_k)$$

de Morgan's law

x_i: i^{th} move of 1^{st} player

y_j: j^{th} move of 2^{nd} player

$$\neg 2^{nd} \ \textit{player has winning strategy}$$

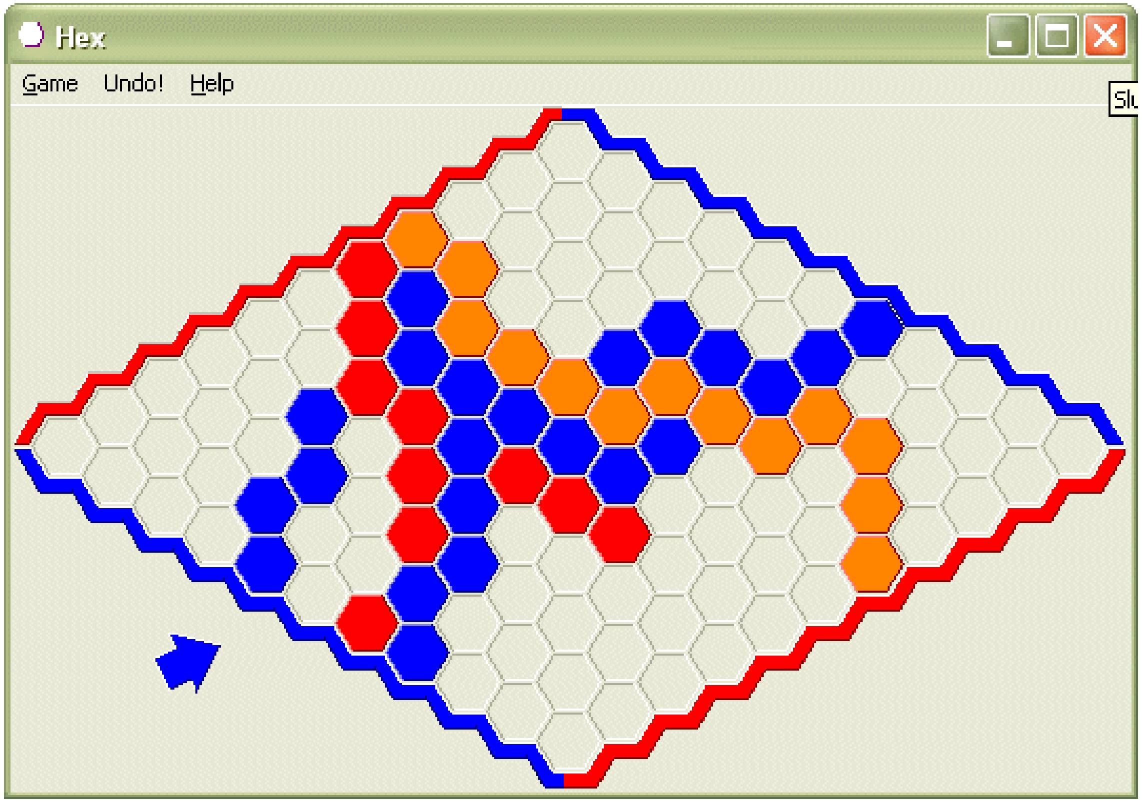
$$\Leftrightarrow \neg \forall x_1 \exists y_1 \cdots \forall x_k \exists y_k \left(2^{nd} \ \textit{player wins}\right)$$

$$\Leftrightarrow \exists x_1 \forall y_1 \cdots \exists x_k \forall y_k \neg \left(2^{nd} \ \textit{player wins}\right)$$

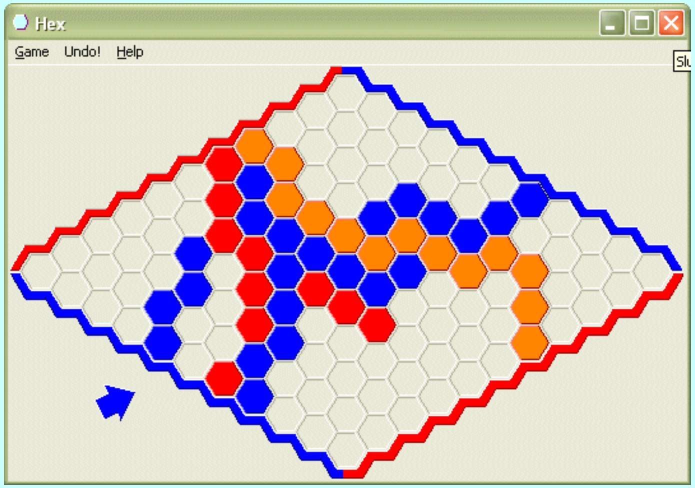
$$\Leftrightarrow \exists x_1 \forall y_1 \cdots \exists x_k \forall y_k \left(1^{st} \ \textit{player wins}\right)$$

$$\Leftrightarrow 1^{st} \ \textit{player has winning strategy}$$

Hex



Hex



In the game Hex, the first player has a wining strategy.

Hex

Need to prove three statements:

1. Hex can never end in a draw.
2. Winning strategy exists for one of the players.
3. The first player has a winning strategy.

Hex

Hex can never end in a draw.	Topology
Winning strategy exists for one of the players.	Zermelo's Theorem
The first player has a winning strategy.	Strategy Stealing

Strategy stealing

Suppose each move does no harm to the player who makes the move. Then the second player cannot have a winning strategy.

Examples: Hex, Tic-tac-toe, Gomoku (Five chess).

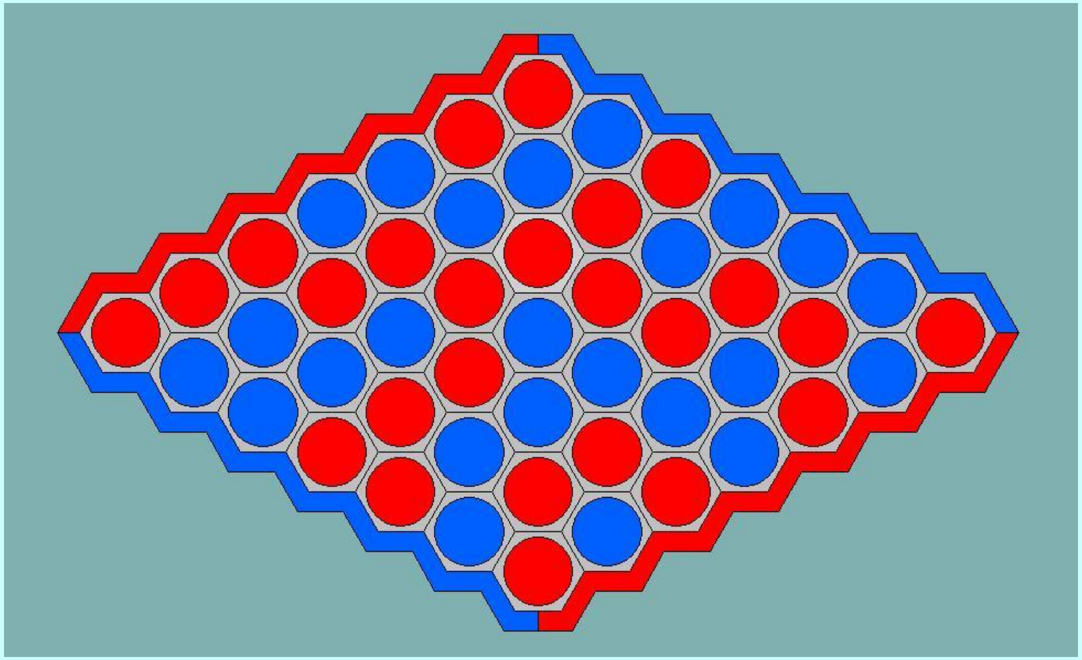
Strategy stealing

Suppose the second player has a winning strategy. The first player could steal it by making an irrelevant first move and then follow the second player's strategy. This ensures a first player win which leads to a contradiction.

Strategy stealing

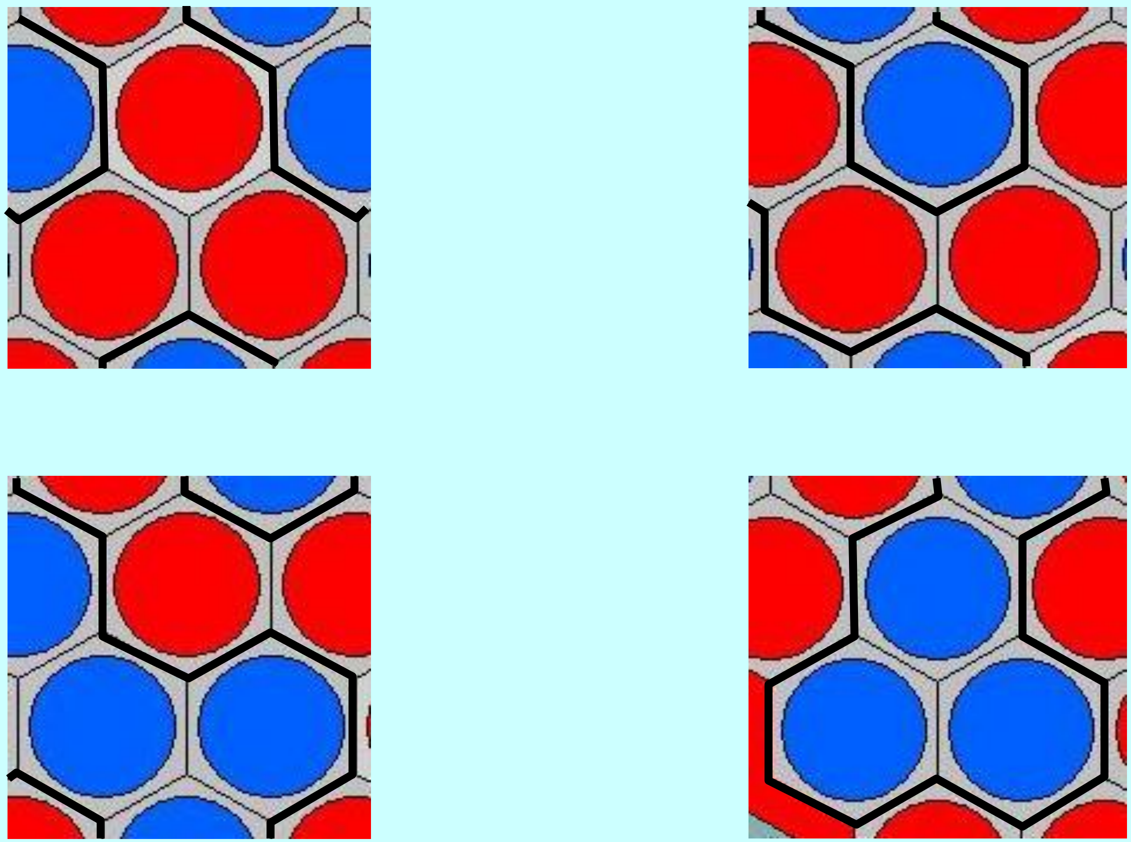
Game	Can end in a Draw	1^{st} player has winning strategy
Hex	No	Yes
Gomoku	Yes	Yes
Tic-tac-toe	Yes	No

Never draw

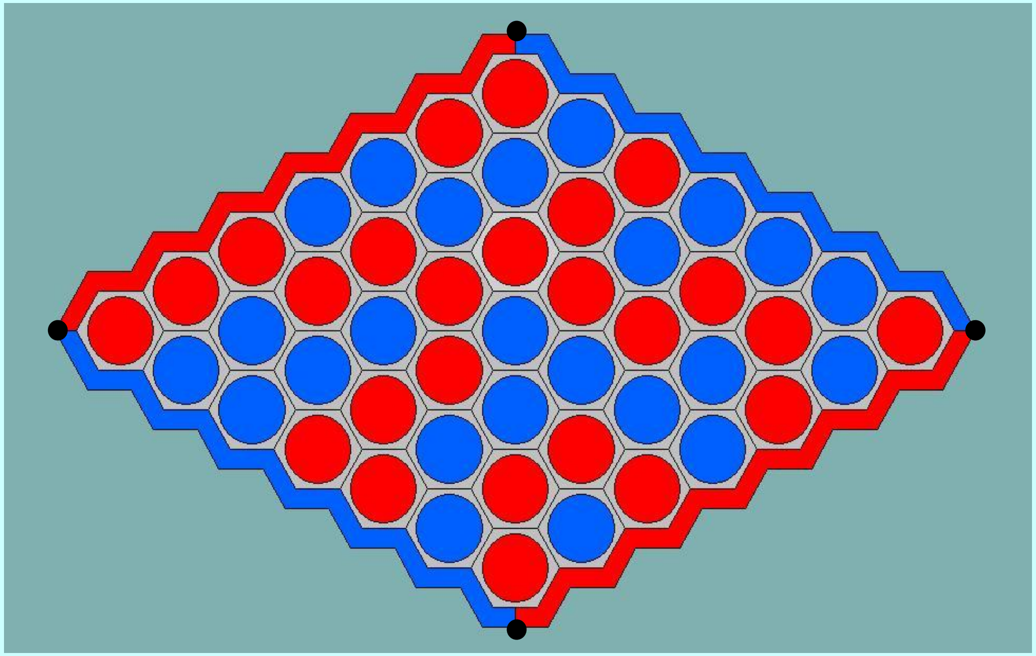


Hex can never end in a draw.

Boundary

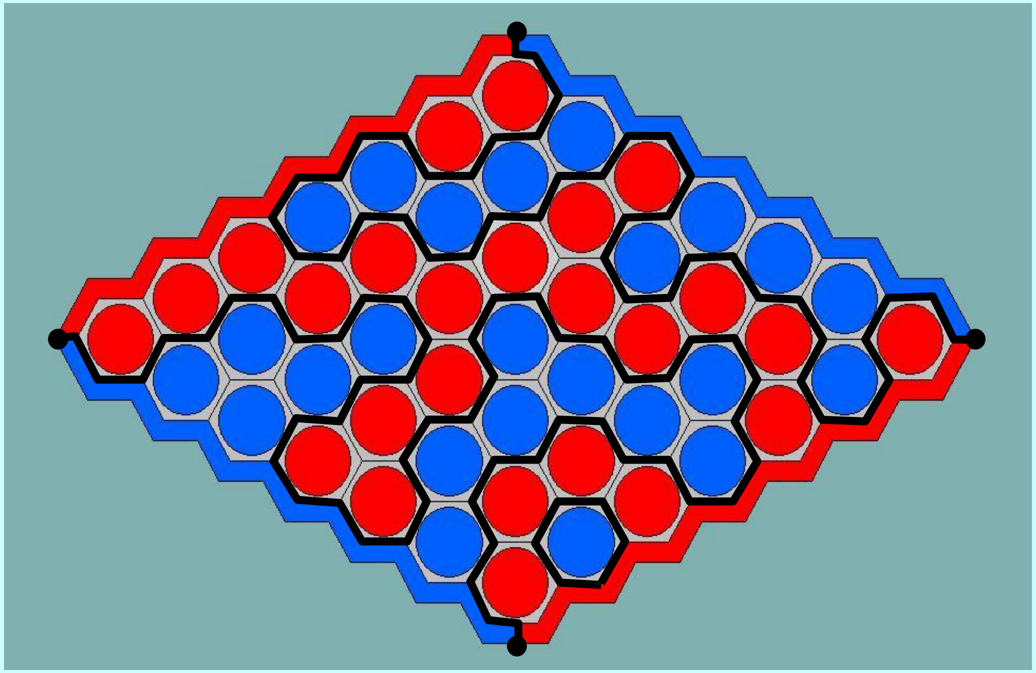


Boundary



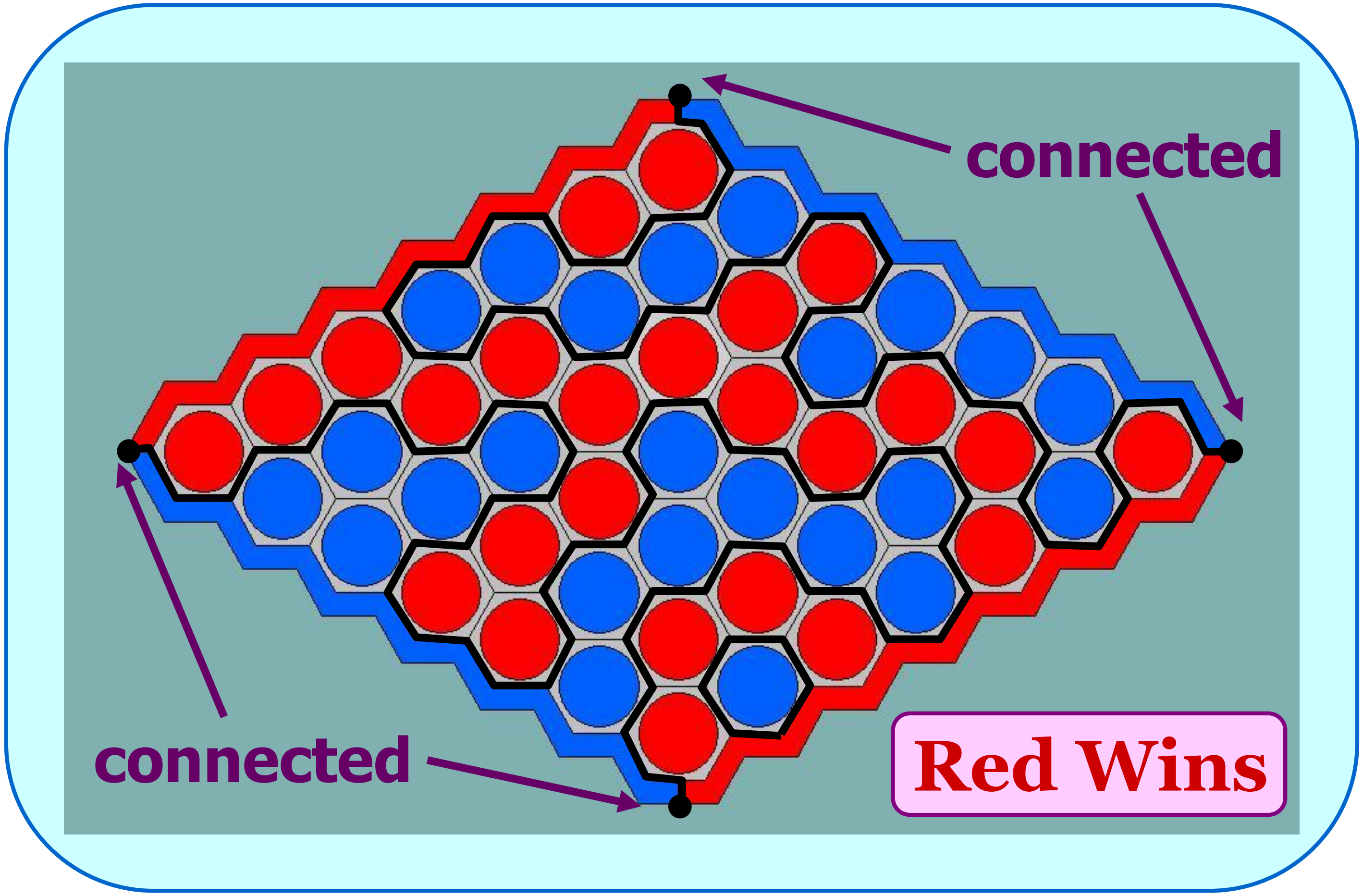
The boundary has no boundary.

Boundary



The boundary has no boundary.

Never draw



Combinatorial games

- How to determine which player has a winning strategy?
- How to find a winning strategy?

P-position and N-position

P-position
The previous player has a winning strategy.

N-position
The next player has a winning strategy.

P-position and N-position

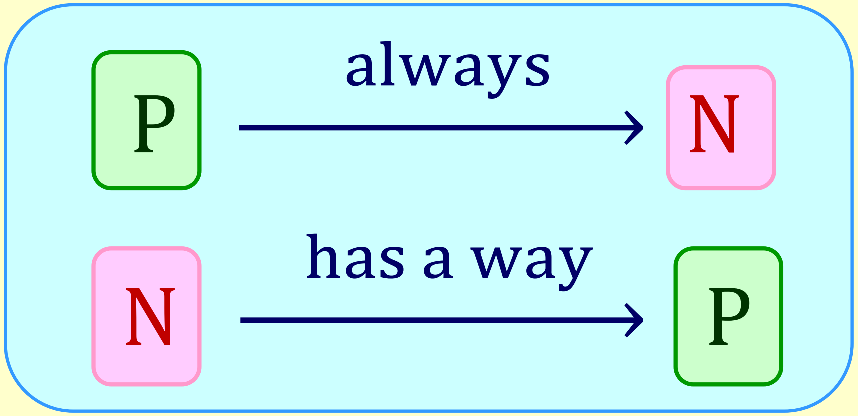
In normal play rule, the player makes the last move wins. In this case,

1. Every terminal position is a P-position
2. A position which can move to a P-position is an N-position
3. A position which can only move to an N-position is a P-position

P-position and N-position

P: previous player has winning strategy

N: next player has winning strategy



Combinatorial games

Q. How to determine which player has a winning strategy?

A. Player with winning strategy for different initial positions
P-position: Second player
N-position: First player

Q. How to find a winning strategy?

A. Keep moving to a P-position.

Take-away game

Take-away game

- There is a pile of *n* chips on the table.
- Two players take turns removing 1, 2, or 3 chips from the pile.
- The player removes the last chip wins.

Take-away game

1. Every terminal position is a P-position

0 1 2 3 4 5 6 7 8 9 10 11 …

P

Take-away game

A position which can move to a P-position is an N-position

0	1	2	3	4	5	6	7	8	9	10	11	…
P	N	N	N									

Take-away game

A position which can only move to an N-position is a P-position

0	1	2	3	4	5	6	7	8	9	10	11	…
P	N	N	N	P								

Take-away game

A position which can move to a P-position is an N-position

0	1	2	3	4	5	6	7	8	9	10	11	…
P	N	N	N	P	N	N	N					

Take-away game

A position which can only move to an N-position is a P-position

0	1	2	3	4	5	6	7	8	9	10	11	…
P	N	N	N	P	N	N	N	P				

Take-away game

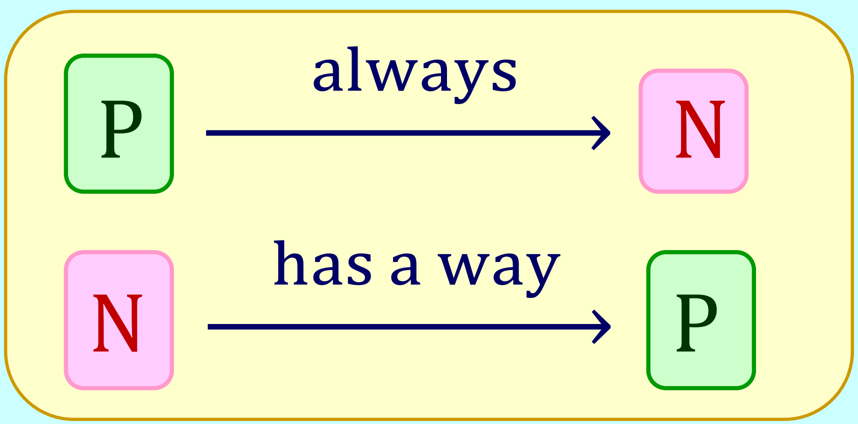
A position which can move to a P-position is an N-position

0	1	2	3	4	5	6	7	8	9	10	11	…
P	N	N	N	P	N	N	N	P	N	N	N	…

Take-away game

P = { 0, 4, 8, 12, 16, 20,… }

N = { not multiple of 4 }



Take-away game

- If the initial position is multiple of 4, the second player has a winning strategy. If the initial position is not a multiple of 4, the first player has a winning strategy.
- A winning strategy is to keep moving to a multiple of 4.

Modified take-away game

Modified take-away game

- There is a pile of *n* chips on the table.
- Two players take turns removing 1, 3, or 4 chips from the pile.
- The player removes the last chip wins.

Modified take-away game

1. Every terminal position is a P-position

0	1	2	3	4	5	6	7	8	9	10	11	…
P												

Modified take-away game

A position which can move to a P-position is an N-position

0	1	2	3	4	5	6	7	8	9	10	11	…
P	N		N	N								

Modified take-away game

A position which can only move to an N-position is a P-position

0	1	2	3	4	5	6	7	8	9	10	11	…
P	N	P	N	N								

Modified take-away game

A position which can move to a P-position is an N-position

0	1	2	3	4	5	6	7	8	9	10	11	…
P	N	P	N	N	N	N						

Modified take-away game

A position which can only move to an N-position is a P-position

0	1	2	3	4	5	6	7	8	9	10	11	…
P	N	P	N	N	N	N	P					

Modified take-away game

A position which can move to a P-position is an N-position

0	1	2	3	4	5	6	7	8	9	10	11	…
P	N	P	N	N	N	N	P	N		N	N	

Modified take-away game

A position which can move to a P-position is an N-position

0	1	2	3	4	5	6	7	8	9	10	11	…
P	N	P	N	N	N	N	P	N	P	N	N	

Modified take-away game

P = { 0, 2, 7, 9, 14, 16,… }

= {*k*: The remainder is 0 or 2 when *k* is divided by 7}

N = { 1, 3, 4, 5, 6, 8, 10, 11,…}

= {*k*: The remainder is 1, 3, 4, 5, 6 when k is divided by 7}

Two piles take-away game

- There are 2 piles of chips
- On each turn, the player may either
  (a) remove any number of chips
  from one of the piles or
  (b) remove the same number of chips
  from both piles.
- The player who removes the last chip wins.

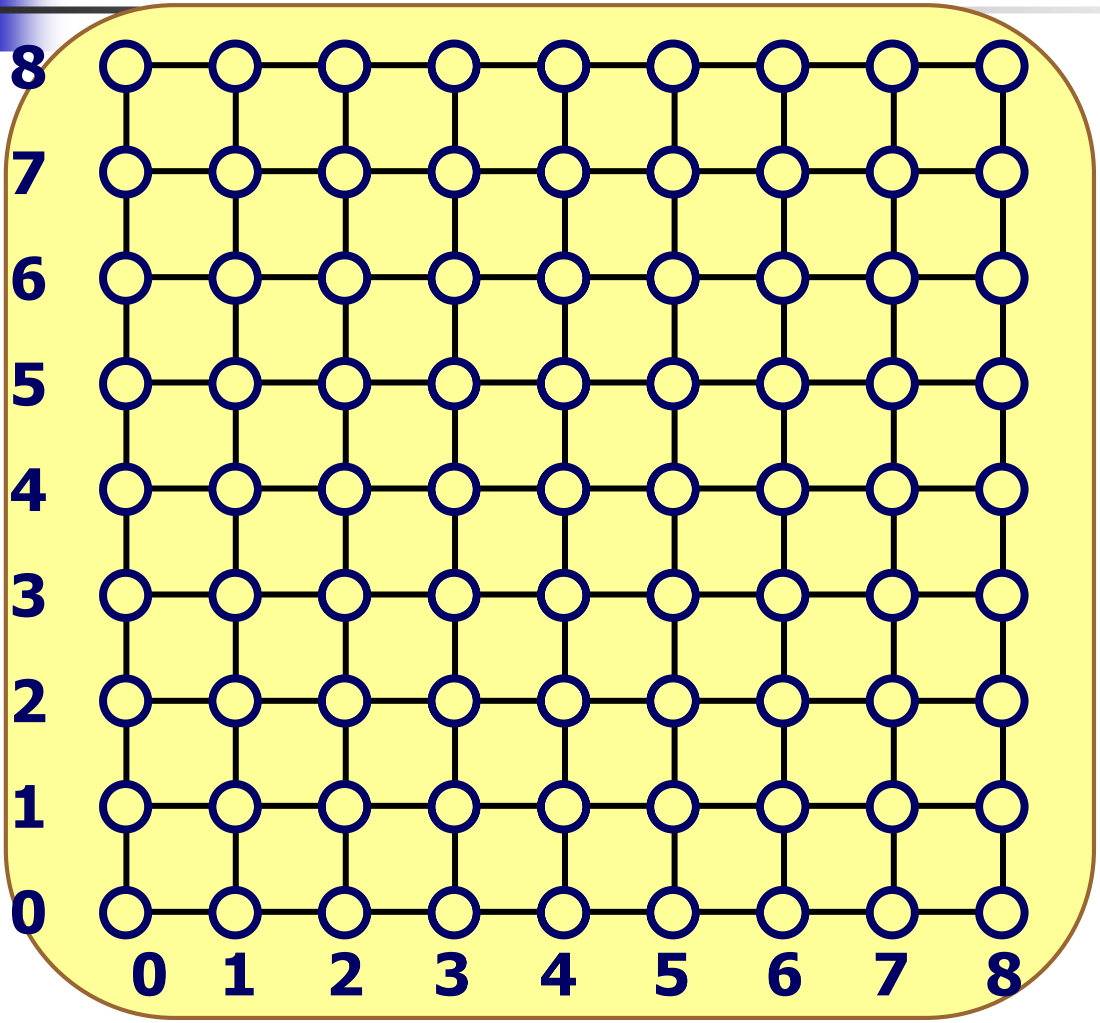
Two piles take-away game

P-positions:

{ (0,0), (1,2), (3,5), ?,… }

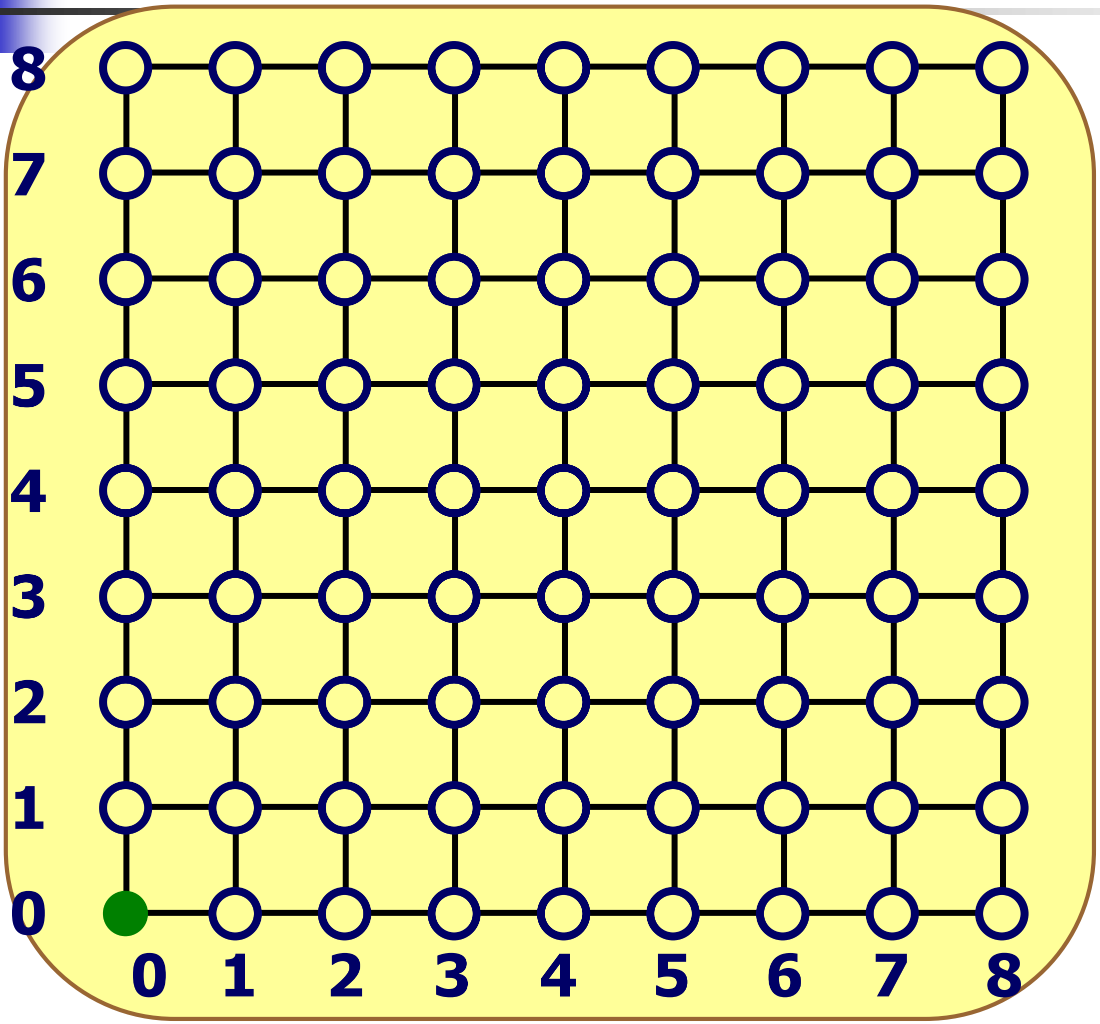
What is the next pair?

Two piles take-away game



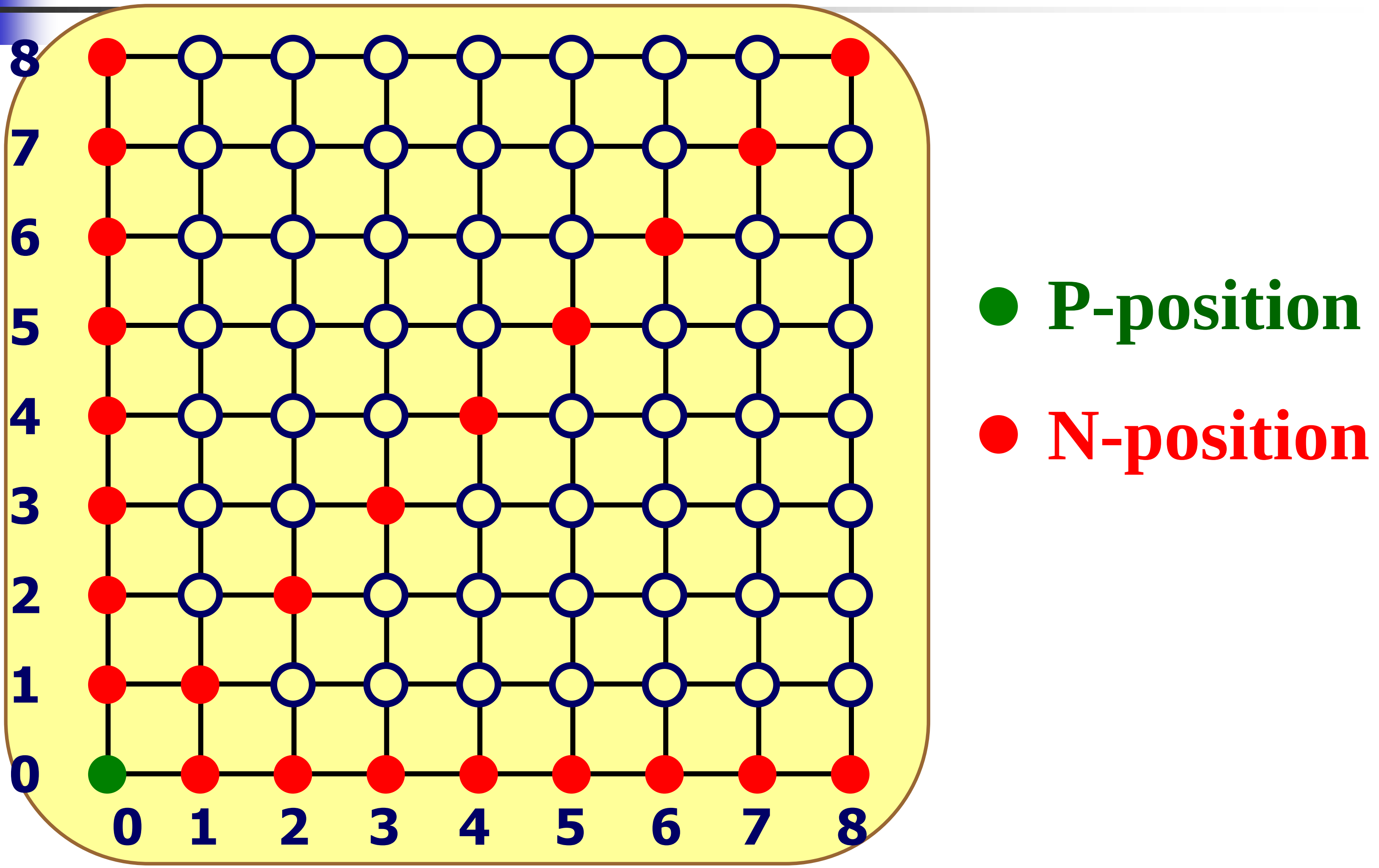
- **P-position**
- **N-position**

Terminal positions are P-positions

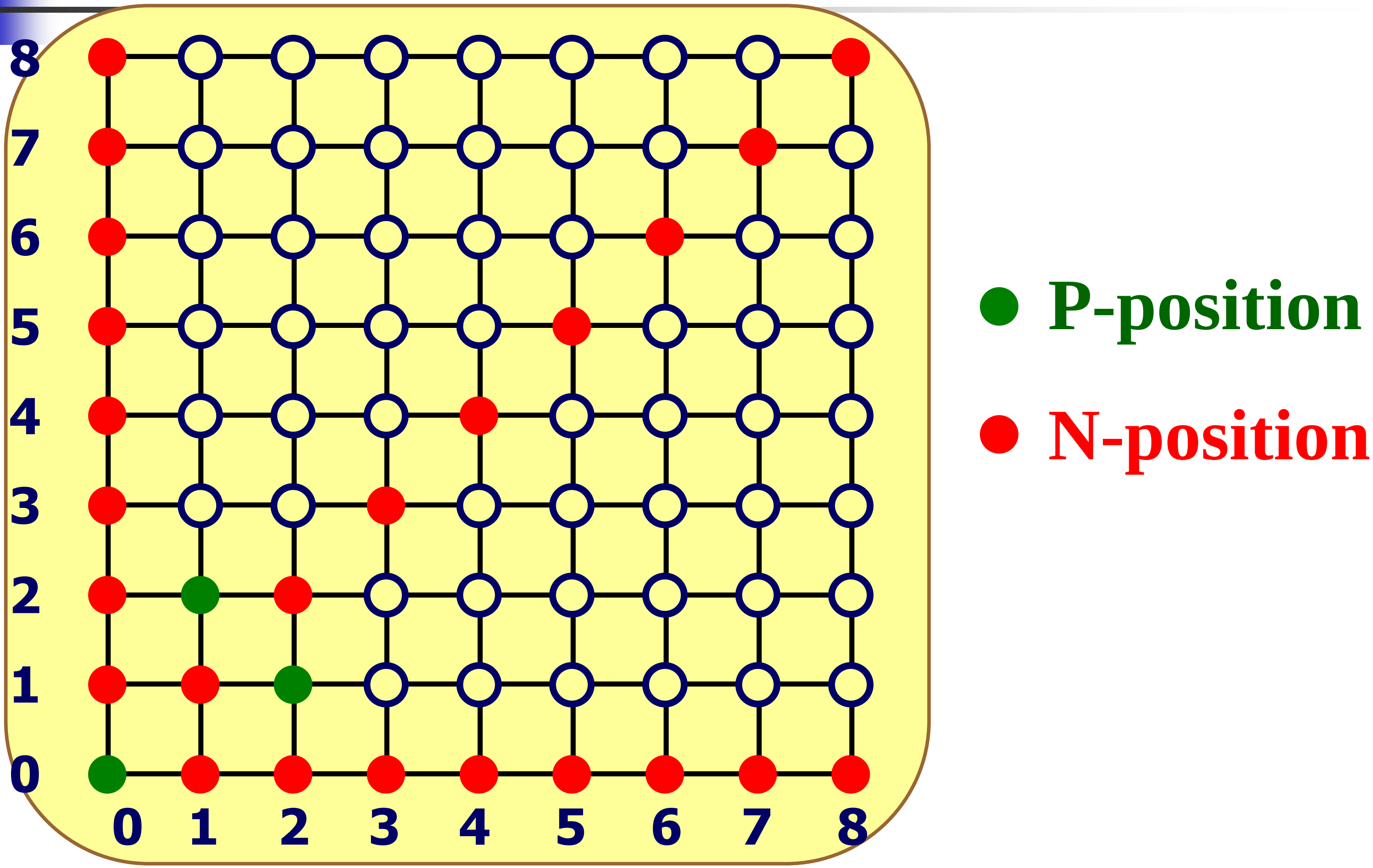


- P-position
- N-position

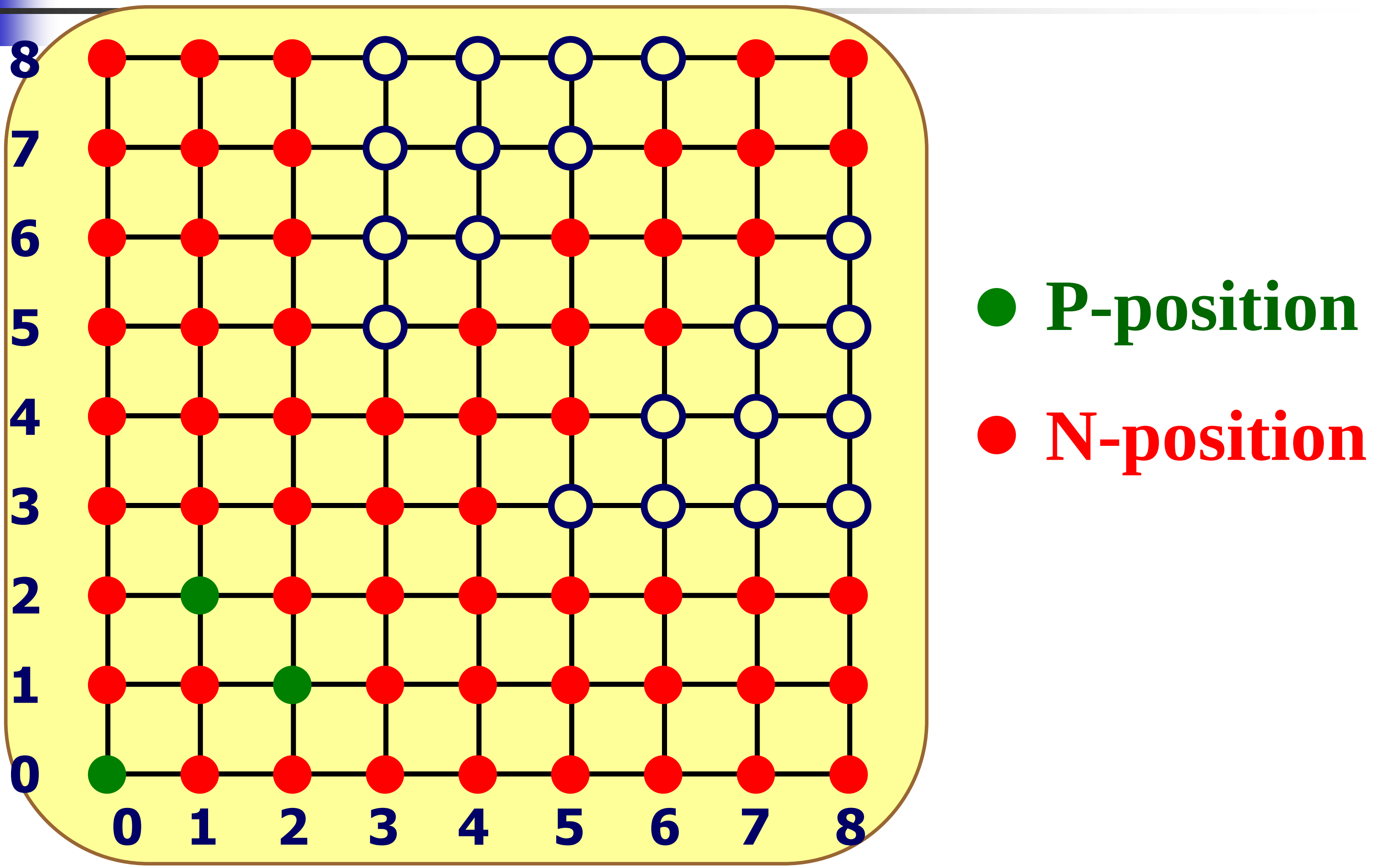
Positions which can move to P-positions are N-positions



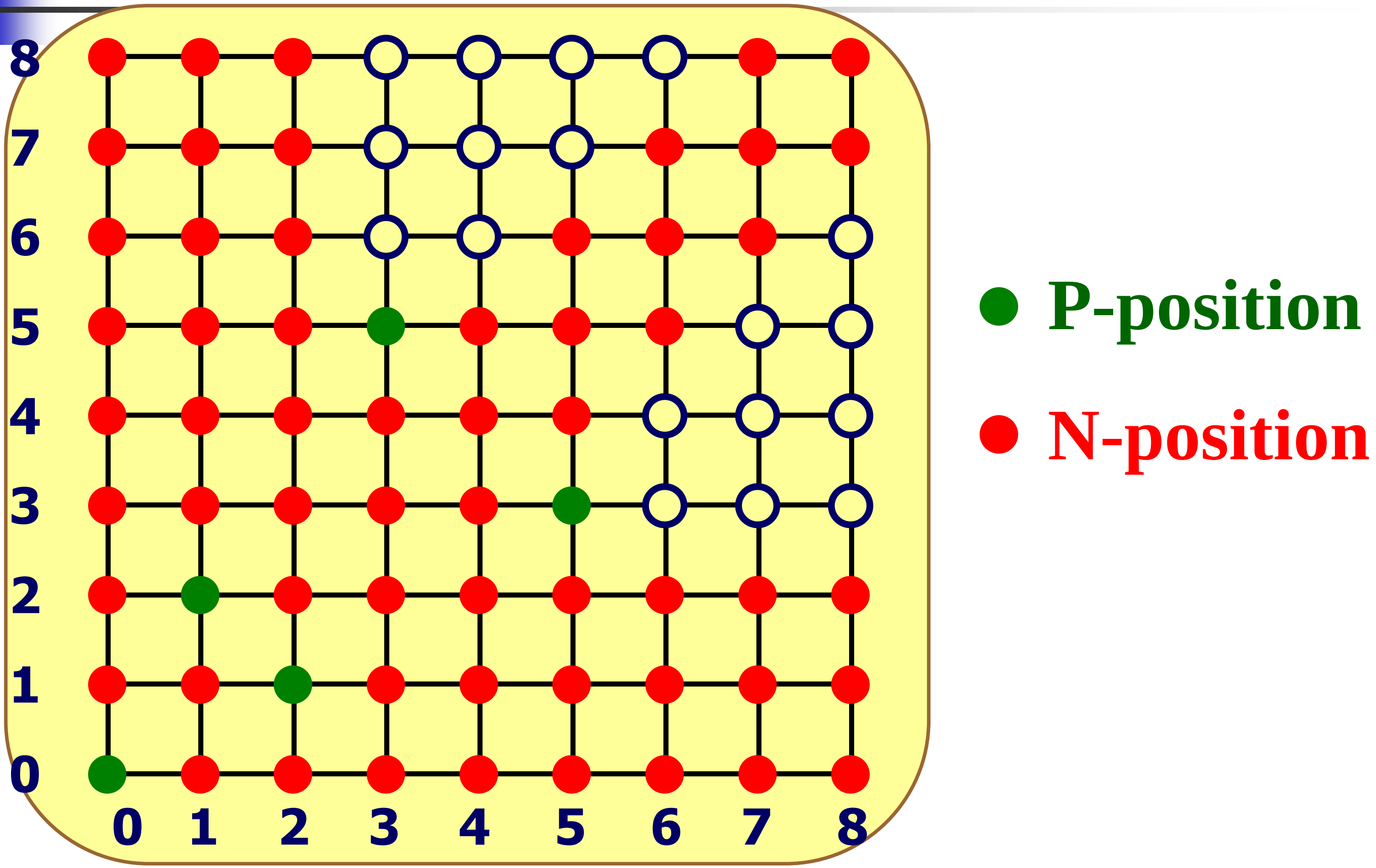
Positions which can only move to N-positions are P-positions



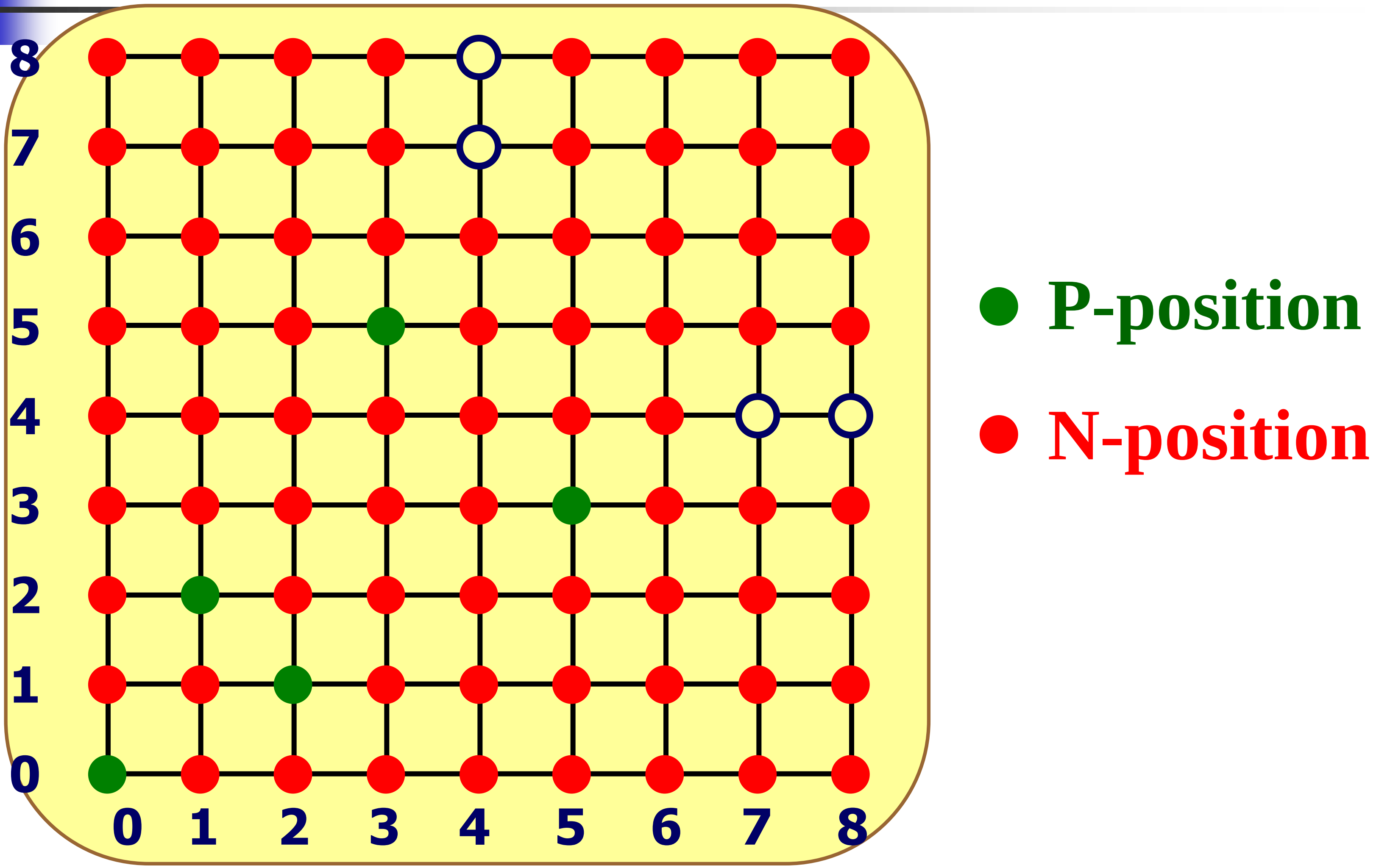
Positions which can move to P-positions are N-positions



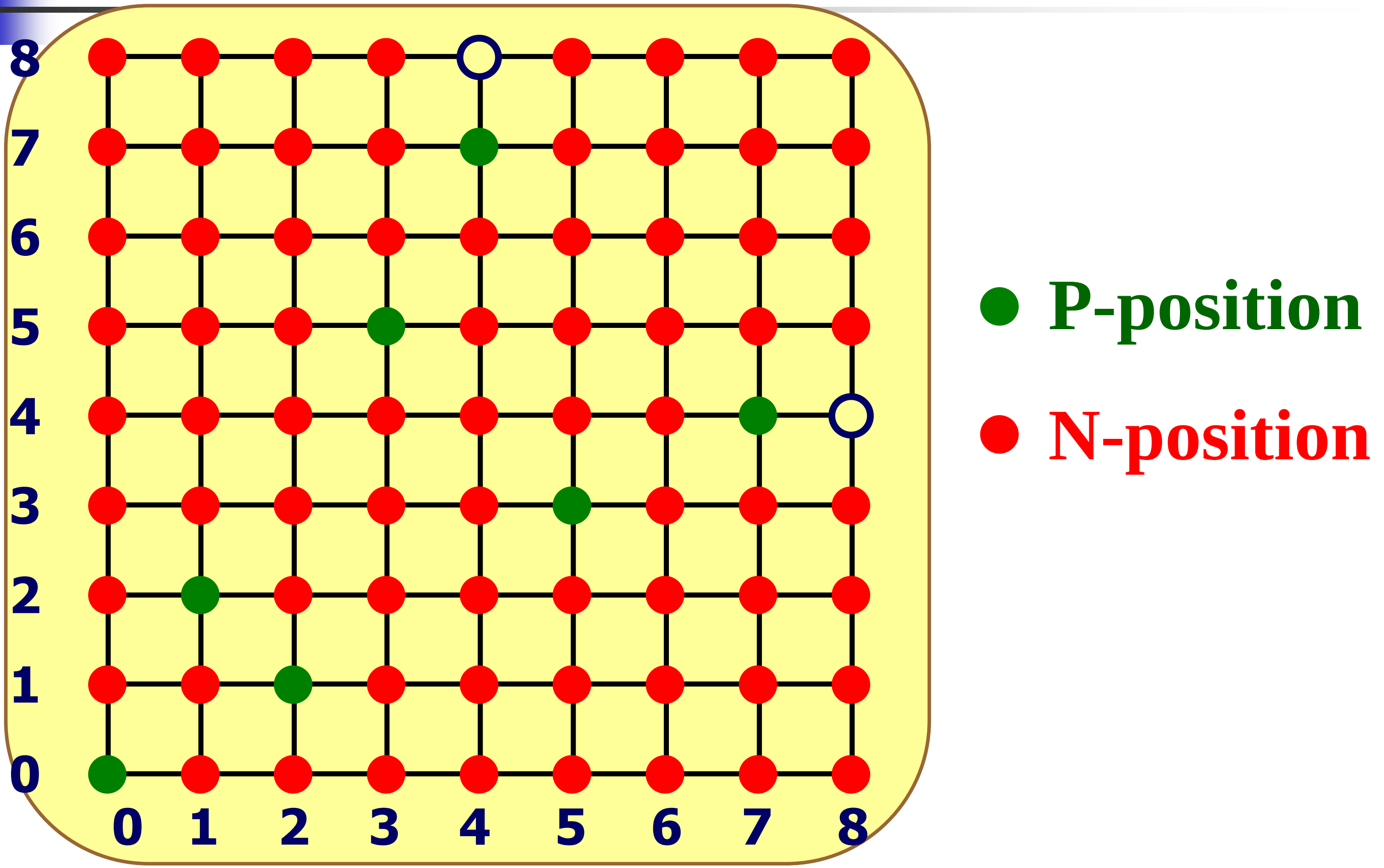
Positions which can only move to N-positions are P-positions



Positions which can move to P-positions are N-positions



Positions which can only move to N-positions are P-positions



Two piles take-away game

(1,2) (3,5) (4,7) (6,10) ?

Two piles take-away game

(1,2) (3,5) (4,7) (6,10) (8,13) …

1. Every integer appears exactly once.

2. The *n*-th pair is different by *n*.

Fibonacci sequence and golden ratio

1, 1, 2, 3, 5, 8, 13, 21, 34, 55,...

Golden ratio:

$$\varphi = \frac{1+\sqrt{5}}{2} \approx 1.6180339887...$$

Golden ratio

n	1	2	3	4	5	6	7
$n\varphi$	1.61	3.23	4.85	6.47	8.09	9.70	11.3
a_n	1	3	4	6	8	9	11
b_n	2	5	7	10	13	15	18

Two piles take-away game

The n^{th} pair is

$$(a_n, b_n) = ([n\varphi], [n\varphi] + n)$$

where $[x]$ is the largest integer not larger than x. In other words, $[x]$ is the unique integer such that

$$x - 1 < [x] \leq x$$

Two piles take-away game

It is easy the see that the n-th pair satisfies

$$b_n - a_n = n$$

To prove that every positive integer appears in the sequences exactly once, observe that

$$\frac{1}{\varphi} + \frac{1}{\varphi + 1} = \frac{2}{1+\sqrt{5}} + \frac{2}{3+\sqrt{5}} = 1$$

and apply the Beatty's theorem.

Beatty's theorem

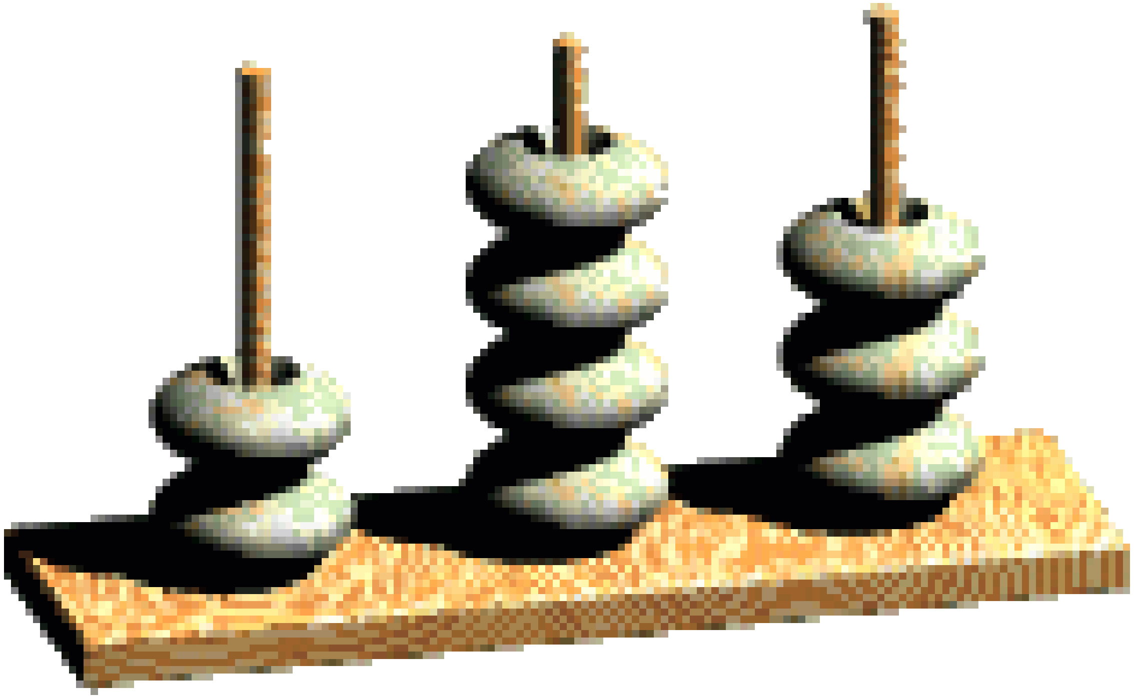
Suppose α and β are positive irrational numbers such that.

$$\frac{1}{\alpha}+\frac{1}{\beta}=1$$

Then every positive integer appears exactly once in the sequences

$$[\alpha], \quad [2\alpha], \quad [3\alpha], \quad [4\alpha], \quad [5\alpha], \cdots$$
$$[\beta], \quad [2\beta], \quad [3\beta], \quad [4\beta], \quad [5\beta], \cdots$$

Nim



Nim

There are three piles of chips. On each turn , the player may

remove any number of chips from any one of the piles.

The player who removes the last chip wins.

Nim

We will use (*x*,*y*,*z*) to represent the position that there are *x*,*y*,*z* chips in the three piles respectively.

Nim

It is easy to see that $(x,x,0)$ is at P-position, in other words the previous player has a winning strategy. By symmetry, $(x,0,x)$ and $(0,x,x)$ are also at P-position.

Nim

By try and error one may also find the following P-positions: (1,2,3), (1,4,5), (1,6,7), (1,8,9), (2,4,6), (2,5,7), (2,8,10), (3,4,7), (3,5,6), (3,8,11),…

Nim

Binary expression:

Decimal	Binary	Decimal	Binary
1	1_2	7	111_2
2	10_2	8	1000_2
3	11_2	9	1001_2
4	100_2	10	1010_2
5	101_2	11	1011_2
6	110_2	12	1100_2

Nim

Nim-sum:
Sum of binary numbers without carry digit.

Examples:

1. $7 \oplus 5 = 2$

$$
\begin{array}{rl}
 & 111_2 = 7 \\
\oplus & 101_2 = 5 \\
\hline
 & 10_2 = 2
\end{array}
$$

Nim

Nim-sum:
Sum of binary numbers without carry digit.

Examples:

2. $23 \oplus 13 = 26$

$$
\begin{array}{rl}
 & 10111_2 = 23 \\
\oplus & 1101_2 = 13 \\
\hline
 & 11010_2 = 26
\end{array}
$$

Nim

Properties:

1. (Associative) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$
2. (Commutative) $x \oplus y = y \oplus x$
3. (Identity) $x \oplus 0 = 0 \oplus x = x$
4. (Inverse) $x \oplus x = 0$
5. (Cancellation law) $x \oplus y = x \oplus z \Rightarrow y = z$

Nim

The position (x,y,z) is at P-position if and only if

$$x \oplus y \oplus z = 0$$

Nim

P-positions:

decimal	(1,2,3)	(1,6,7)	(2,4,6)	(2,5,7)	(3,4,7)
binary	001 010 011	001 110 111	010 100 110	010 101 111	011 100 111

The number of 1's in each column is even (either 0 or 2).

Nim

Examples:

1. (7,5,3)

$$7 \oplus 5 \oplus 3 = 1 \neq 0$$

$$\begin{array}{rr} & 111_2 = 7 \\ & 101_2 = 5 \\ \oplus & 11_2 = 3 \\ \hline & 1_2 = 1 \end{array}$$

It is at N-position. Next player may win by removing 1 chip from any pile and reach P-positions (6,5,3), (7,4,3) or (7,5,2).

Nim

Examples:

2. (25,21,11)

$$25 \oplus 21 \oplus 11 = 7 \neq 0$$

It is at N-position. Next player may win by removing 3 chips from the second pile and reach P-position (25,18,11).

$$\begin{array}{rl} 11001_2 & = 25 \\ 10101_2 & = 21 \\ \oplus \quad 1011_2 & = 11 \\ \hline 111_2 & = 7 \end{array}$$

Nim

Examples:

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$$25 \oplus 21 \oplus 11 = 7 \neq 0$$

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It is at N-position. Next player may win by removing 3 chips from the second pile and reach P-position (25,18,11).

Note: $21 \oplus 7 = 18$

Financial tsunami

Rules:

- The investor may decide the amount of money he uses to buy a fund in each round.
- The return rate in each round is 100% except when “financial tsunami” occurs.
- When the “financial tsunami” occurs, the return rate is -100%.
- “Financial tsunami” will occur at exactly one of the rounds.

Financial tsunami

We may consider the game as a zero sum game between the “Investor” and the “Market”.

Suppose that initially the investor has \$1 and the game is played for n rounds.

Financial tsunami

Suppose the optimal strategy for the investor is to invest $$p_n$ in the first round for some p_n to be determined.

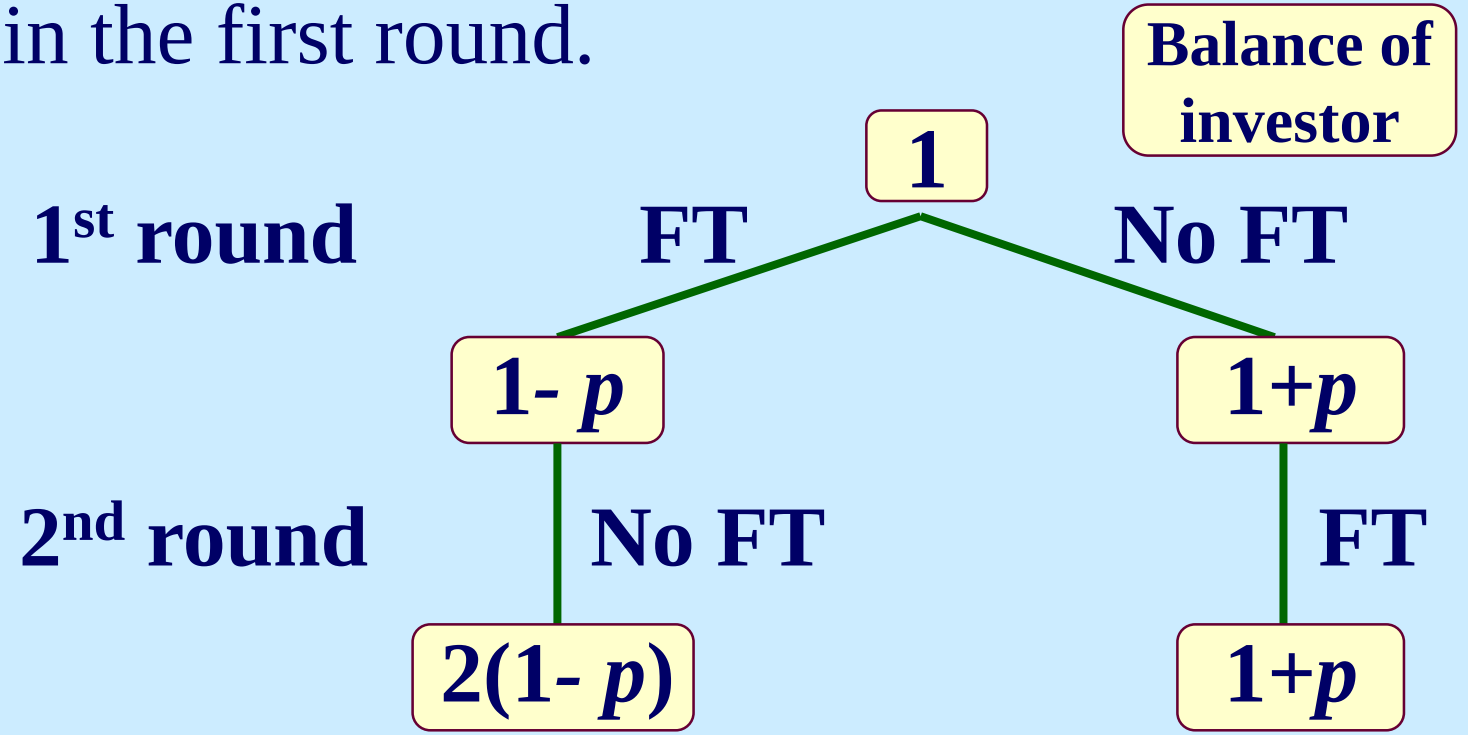
Let $$x_n$ be the balance of the investor after n rounds provided that both the investor and the "Market" use their optimal strategies.

Financial tsunami

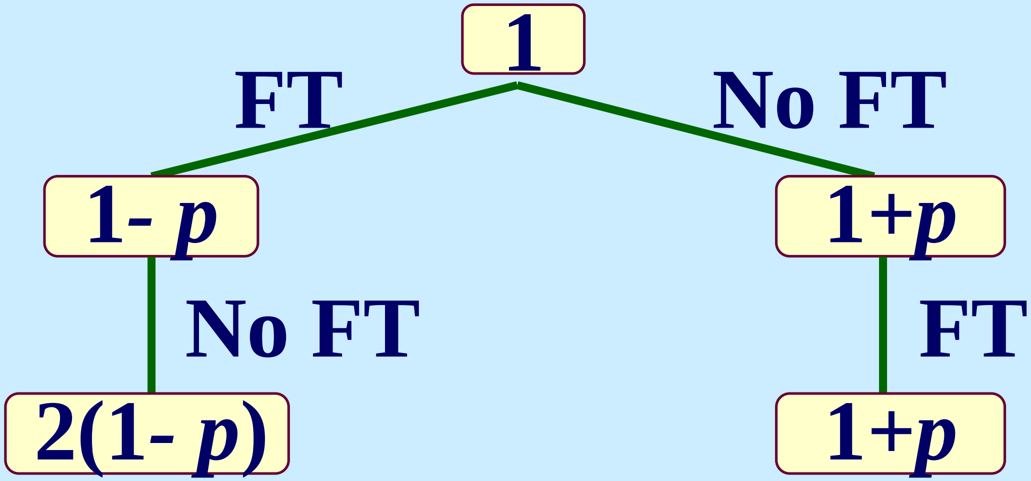
It is obvious that that the investor should invest $0 if there is only 1 round ($n = 1$). Therefore $p_1 = 0$ and $x_1 = 1$.

Financial tsunami

Suppose $n = 2$ and the investor invests \$$p$ in the first round.



Financial tsunami



The optimal strategy for the "Market" is

1. FT in 1st round if $2(1-p) \le 1+p$
2. FT in 2nd round if $1+p \le 2(1-p)$

Financial tsunami

The optimal strategy for the investor is to choose p such that

$$1 + p = 2(1 - p)$$

$$p = \frac{1}{3}$$

Then the balance of investor after 2 rounds is

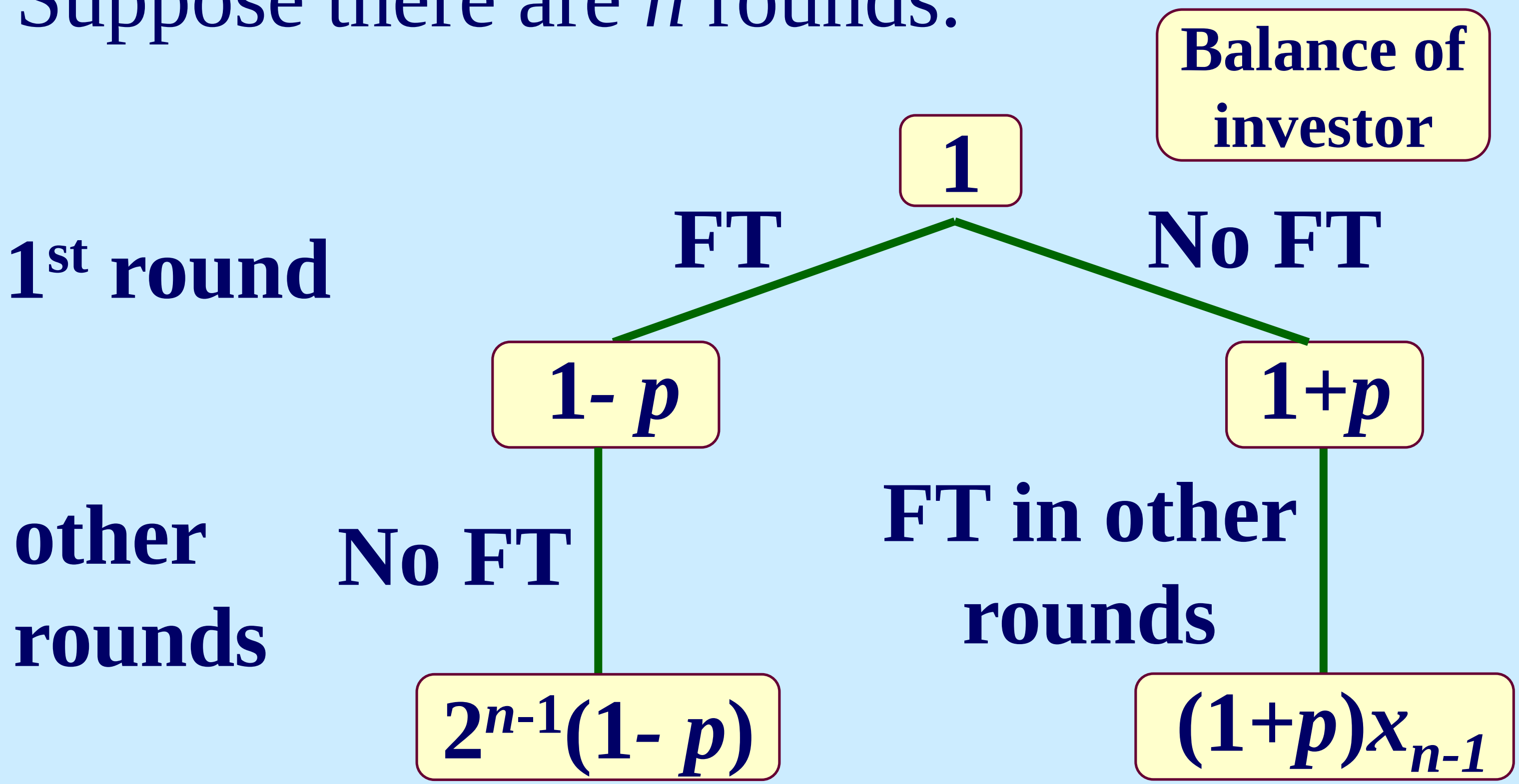
$$1 + \frac{1}{3} = 2\left(1 - \frac{1}{3}\right) = \frac{4}{3}$$

Therefore

$$p_2 = \frac{1}{3} \quad \textbf{and} \quad x_2 = \frac{4}{3}$$

Financial tsunami

Suppose there are n rounds.



Financial tsunami

Similar to the previous argument, p_n and x_n should satisfies

$$x_n = 2^{n-1}(1 - p_n) = (1 + p_n)x_{n-1}$$

Replacing n by n-1 in the first equality, we have

$$x_{n-1} = 2^{n-2}(1 - p_{n-1})$$

Financial tsunami

Substitute it into the second equality, we obtain

$$2^{n-2}\left(1-p_{n-1}\right)\left(1+p_n\right)=2^{n-1}\left(1-p_n\right)$$

Making p_n as the subject, we have

$$1-p_{n-1}+p_n-p_{n-1}p_n=2\left(1-p_n\right)$$

$$p_n=\frac{1+p_{n-1}}{3-p_{n-1}}$$

Financial tsunami

n	p_n
1	0
2	1/3
3	1/2
4	3/5
5	2/3
6	5/7
7	3/4
8	7/9

Financial tsunami

n	p_n
1	0
2	1/3
3	1/2 = 2/4
4	3/5
5	2/3 = 4/6
6	5/7
7	3/4 = 6/8
8	7/9

Financial tsunami

By induction we have

$$p_n = \frac{n-1}{n+1}$$

and

$$x_n = 2^{n-1}\left(1 - p_n\right) = \frac{2^n}{n+1}$$

Financial tsunami

n	p_n	x_n
1	0	1
2	1/3	4/3
3	1/2	2
4	3/5	16/5
5	2/3	16/3
6	5/7	64/7
7	3/4	16

Financial tsunami

Nash equilibrium:
It does not matter when the "Financial Tsunami" occurs.